Principles of Communications
ECS 332

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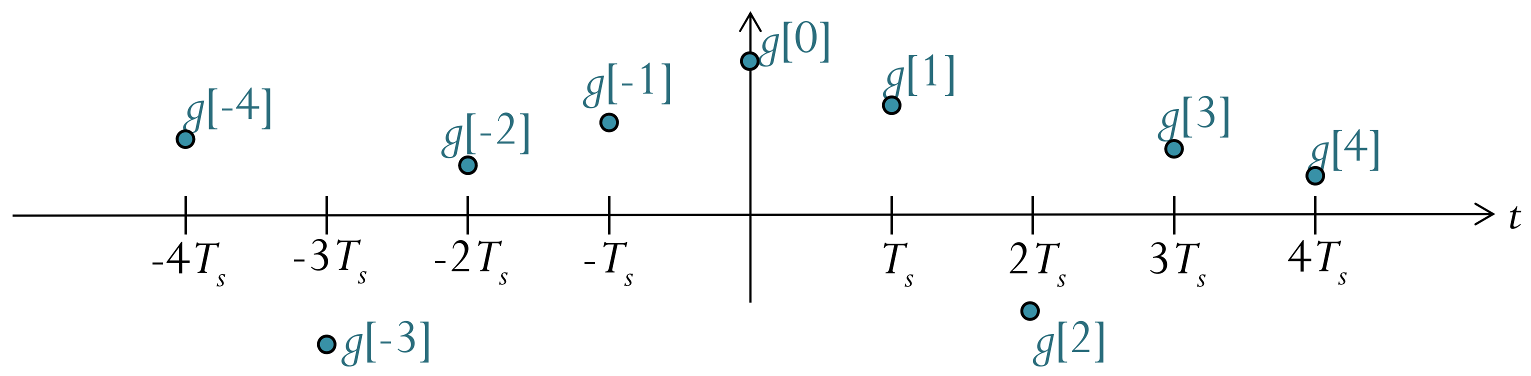
6.2 Ideal Sampling

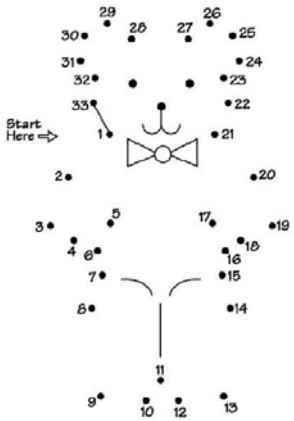
(Section 6.3)

Prelude to Reconstruction

- Start with discrete-time samples.



“Connect-the-Dots” interpolation



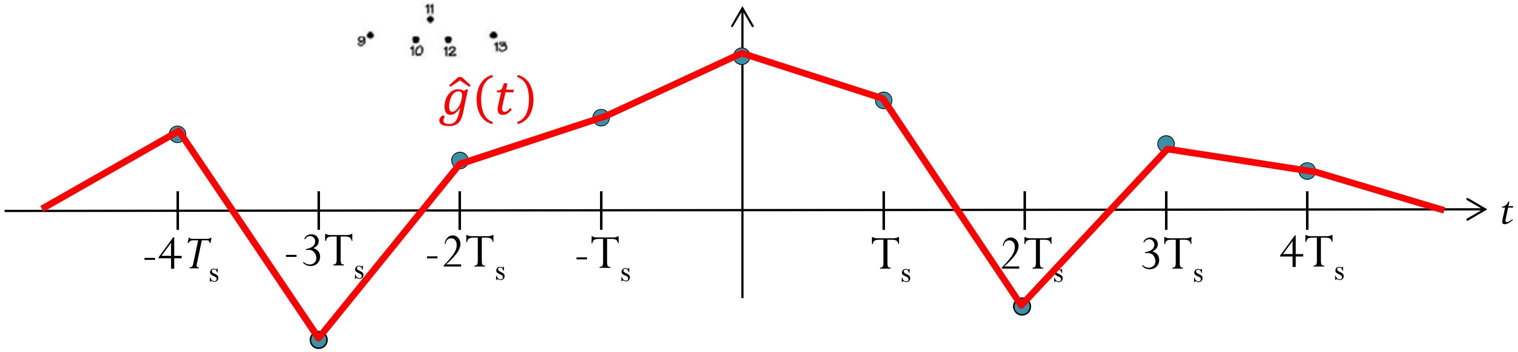


Figure 51

“Connect-the-Dots” interpolation

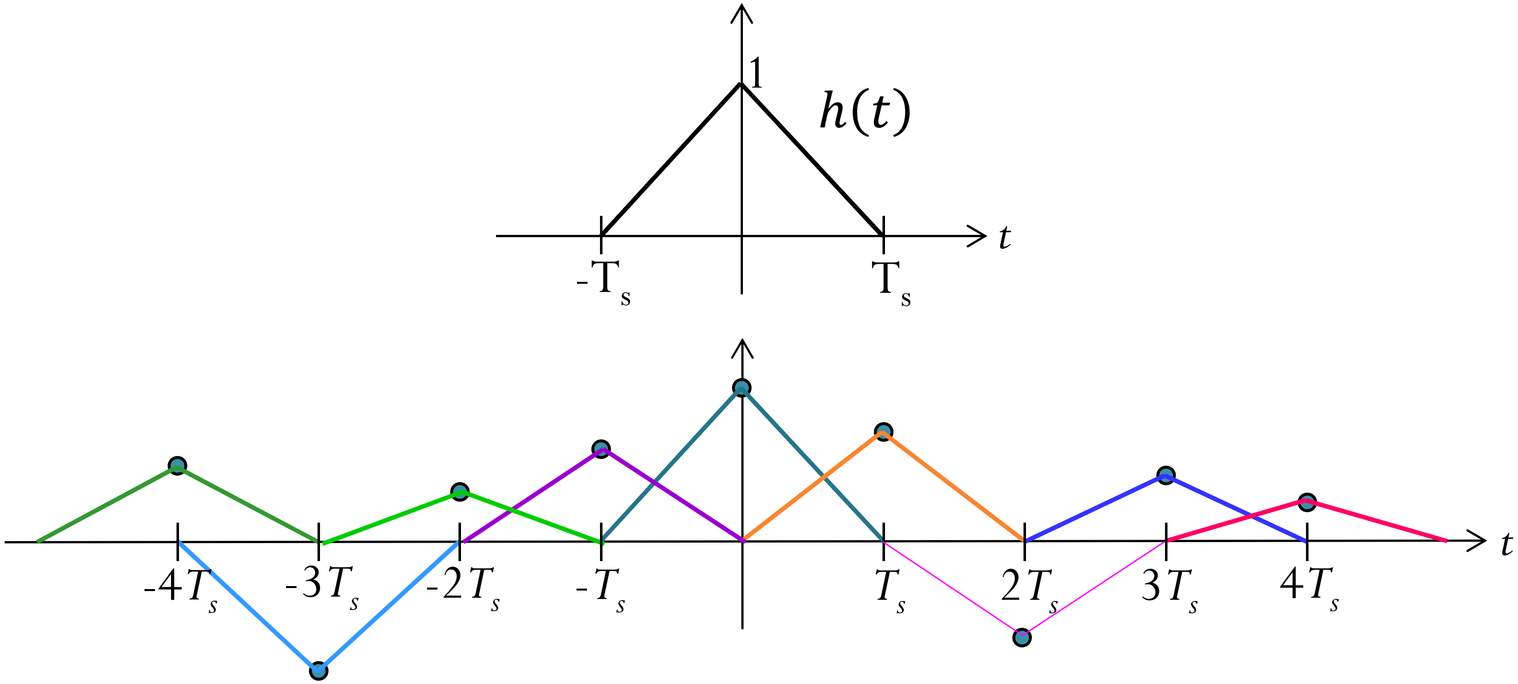


Figure 51

"Connect-the-Dots" interpolation

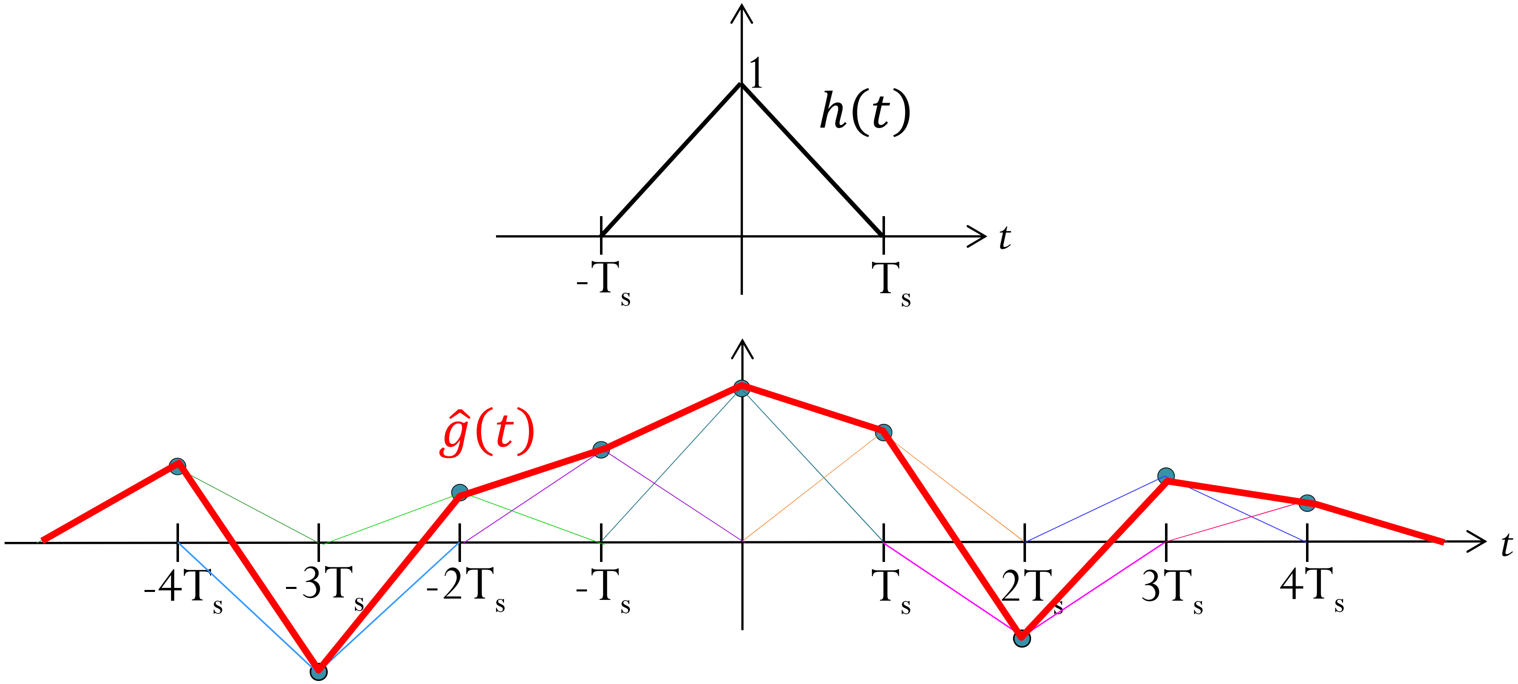
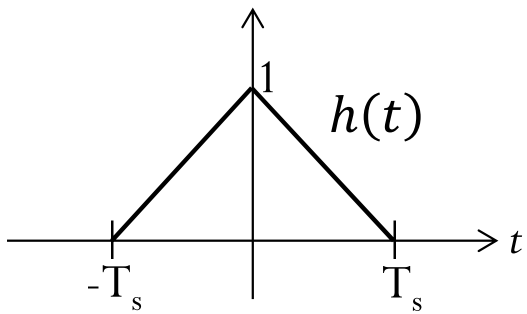
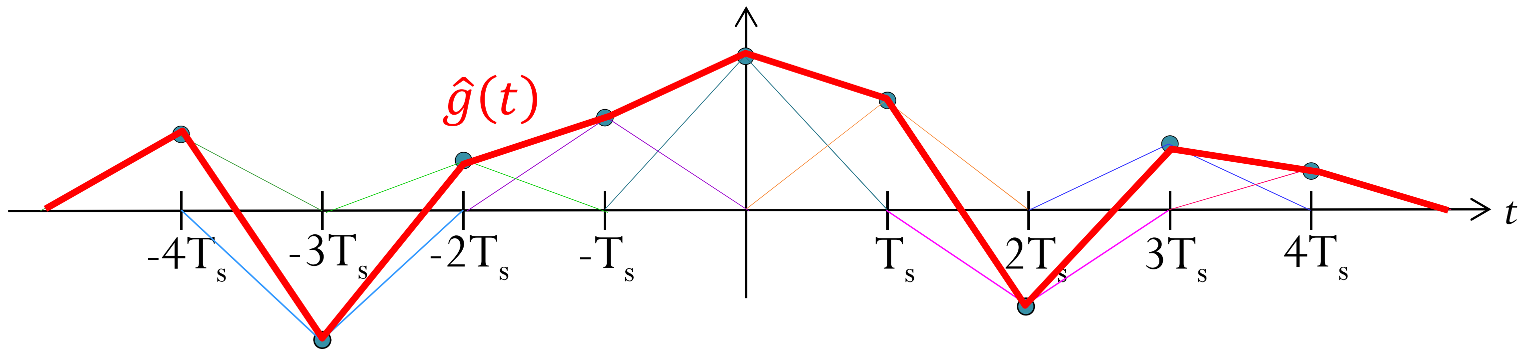


Figure 51

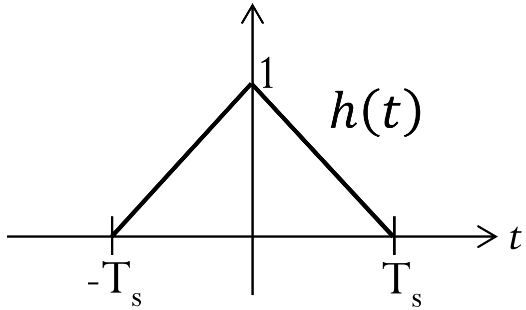
"Connect-the-Dots" interpolation

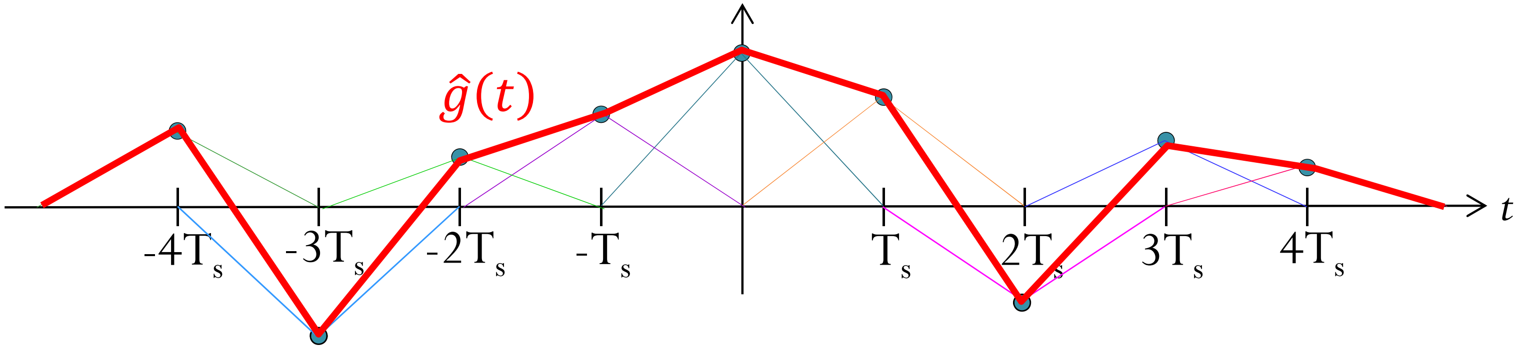




$$\hat{g}(t) = \sum_{n=-\infty}^{\infty} g[n]\, h(t - nT_s)$$

"Connect-the-Dots" interpolation



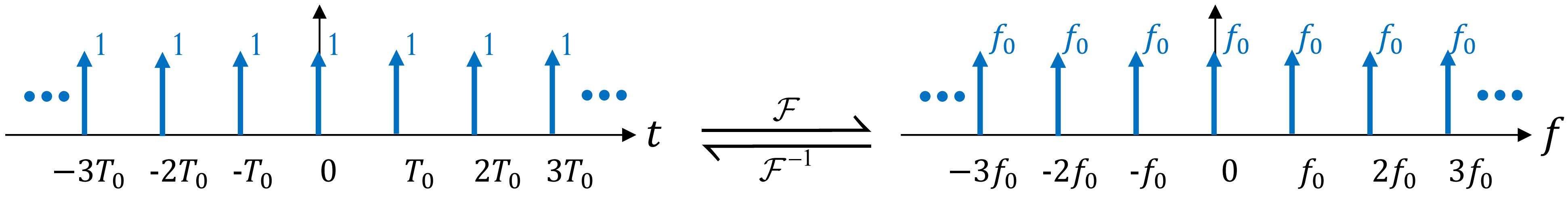


$$\hat{g}(t) = \sum_{n=-\infty}^{\infty} g[n]\, h(t - nT_s) = \sum_{n=-\infty}^{\infty} g[n] \left(\delta(t - nT_s) * h(t) \right)$$

$$= \left(\sum_{n=-\infty}^{\infty} g[n]\, \delta(t - nT_s) \right) * h(t) = g_\delta(t) * h(t)$$

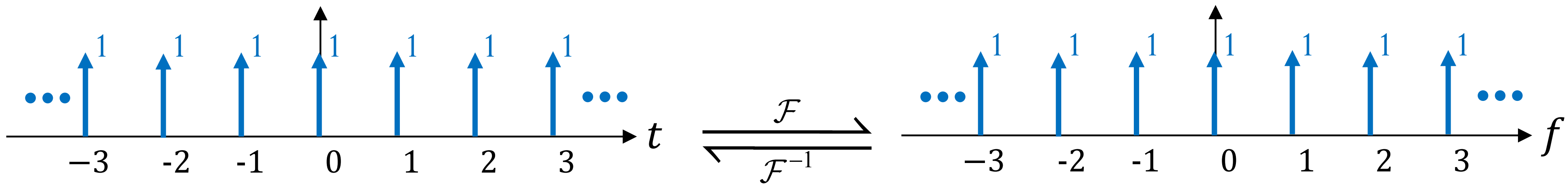
Important Fourier Transform Pair

$$\sum_{n=-\infty}^{\infty}\delta\left(t-nT_0\right)=\sum_{k=-\infty}^{\infty}\frac{1}{T_0}e^{j2\pi(kf_0)t} \qquad f_0\sum_{k=-\infty}^{\infty}\delta\left(t-kf_0\right)$$



In particular,

$$\sum_{n=-\infty}^{\infty}\delta\left(t-n\right) \qquad \sum_{k=-\infty}^{\infty}\delta\left(f-k\right)$$



Making Copies (Replicas, Clones): Modulation vs. Ideal Sampling

- **Modulation**:
  - Let $y(t) = g(t)\cos(2\pi f_c t)$.
  - Then, $Y(f) = \frac{1}{2}G(f - f_c) + \frac{1}{2}G(f + f_c)$.
- **Ideal Sampling**:
  - Let

$$g_\delta(t) = \sum_{n=-\infty}^{\infty} g[n]\,\delta(t - nT_s).$$

    - where $g[n] = g(nT_s)$.
  - Then,

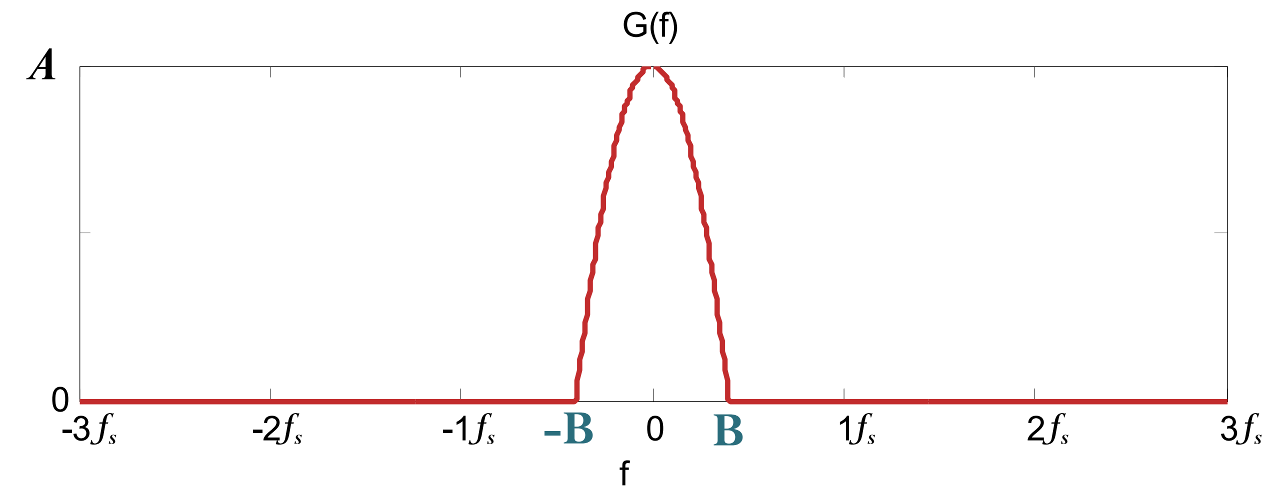
$$G_\delta(f) = f_s \sum_{k=-\infty}^{\infty} G(f - kf_s).$$

    - where $f_s = \frac{1}{T_s}$.

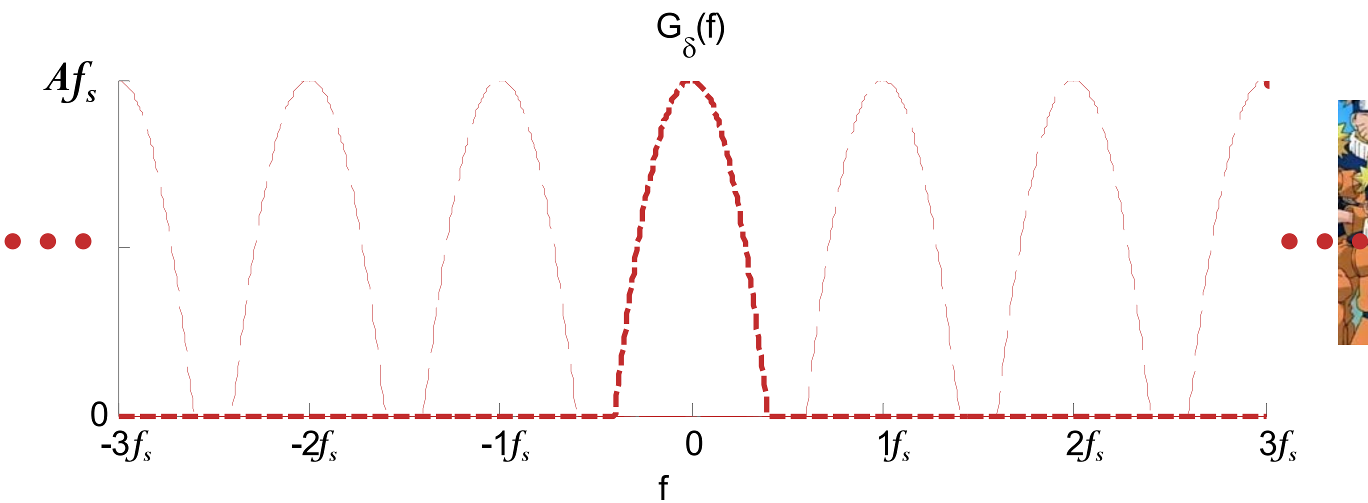
Ideal Sampling

Figure 47

The Fourier transform of the original signal



The Fourier transform of the (ideal) sampled signal

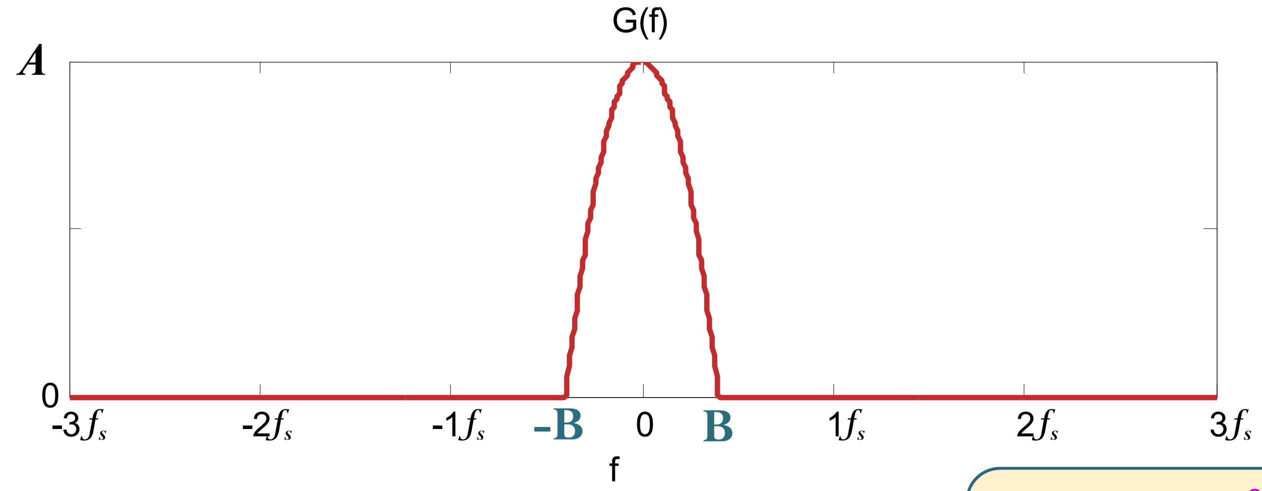


Ideal Sampling

$$g[n] = g(t)|_{t=nT_s} = g(nT_s)$$

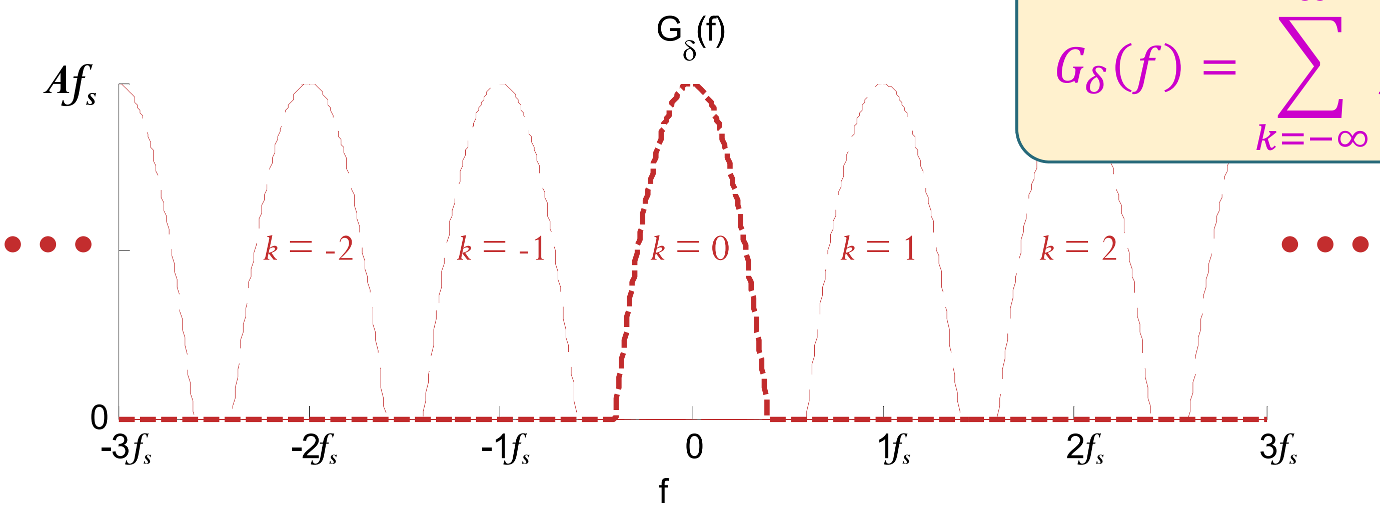
$$g_\delta(t) = \sum_{n=-\infty}^{\infty} g[n]\delta(t - nT_s)$$

The Fourier transform of the original signal



$$G_\delta(f) = \sum_{k=-\infty}^{\infty} f_s G(f - kf_s)$$

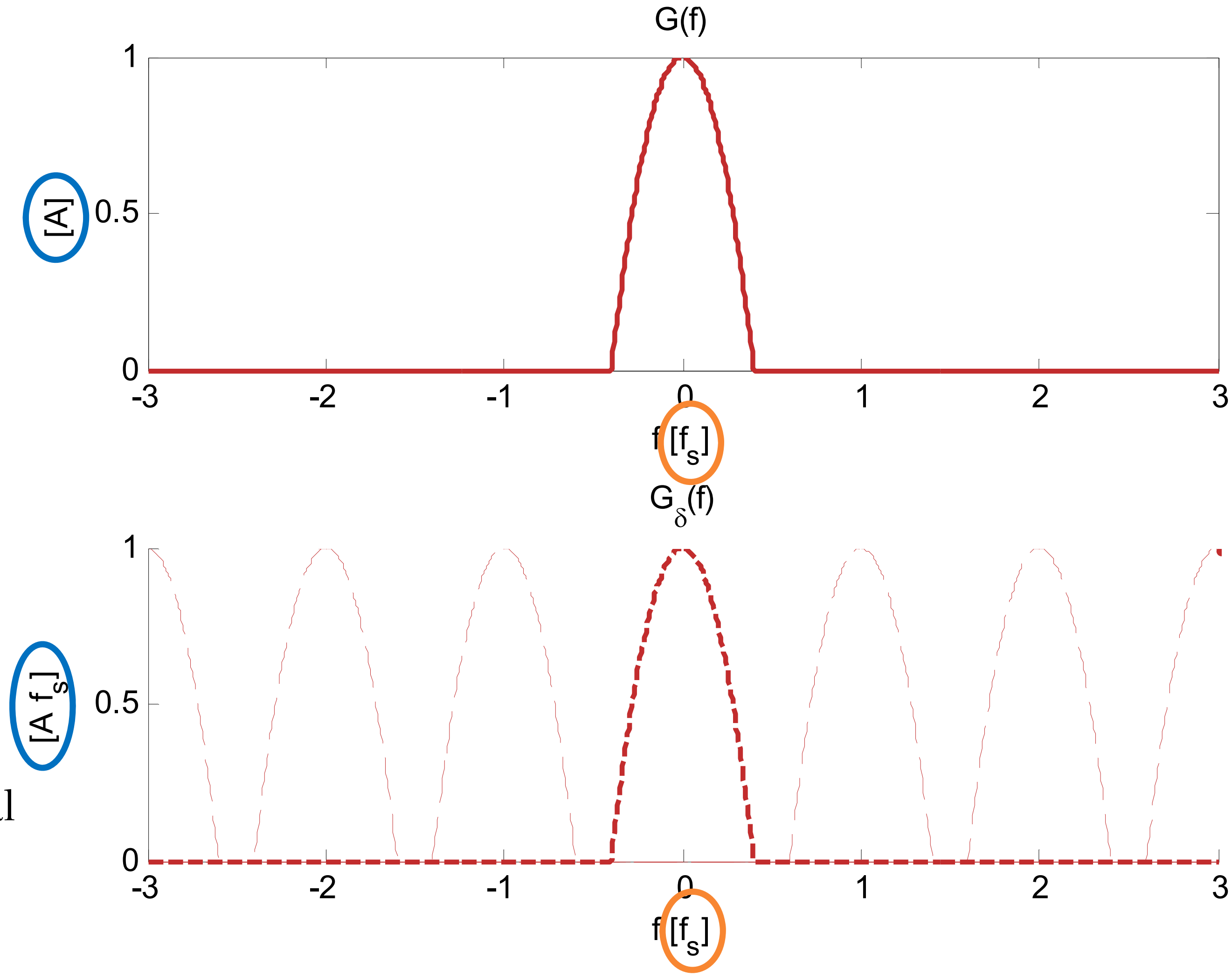
The Fourier transform of the (ideal) sampled signal



Ideal Sampling: MATLAB Exploration

The Fourier transform of the original signal

The Fourier transform of the (ideal) sampled signal

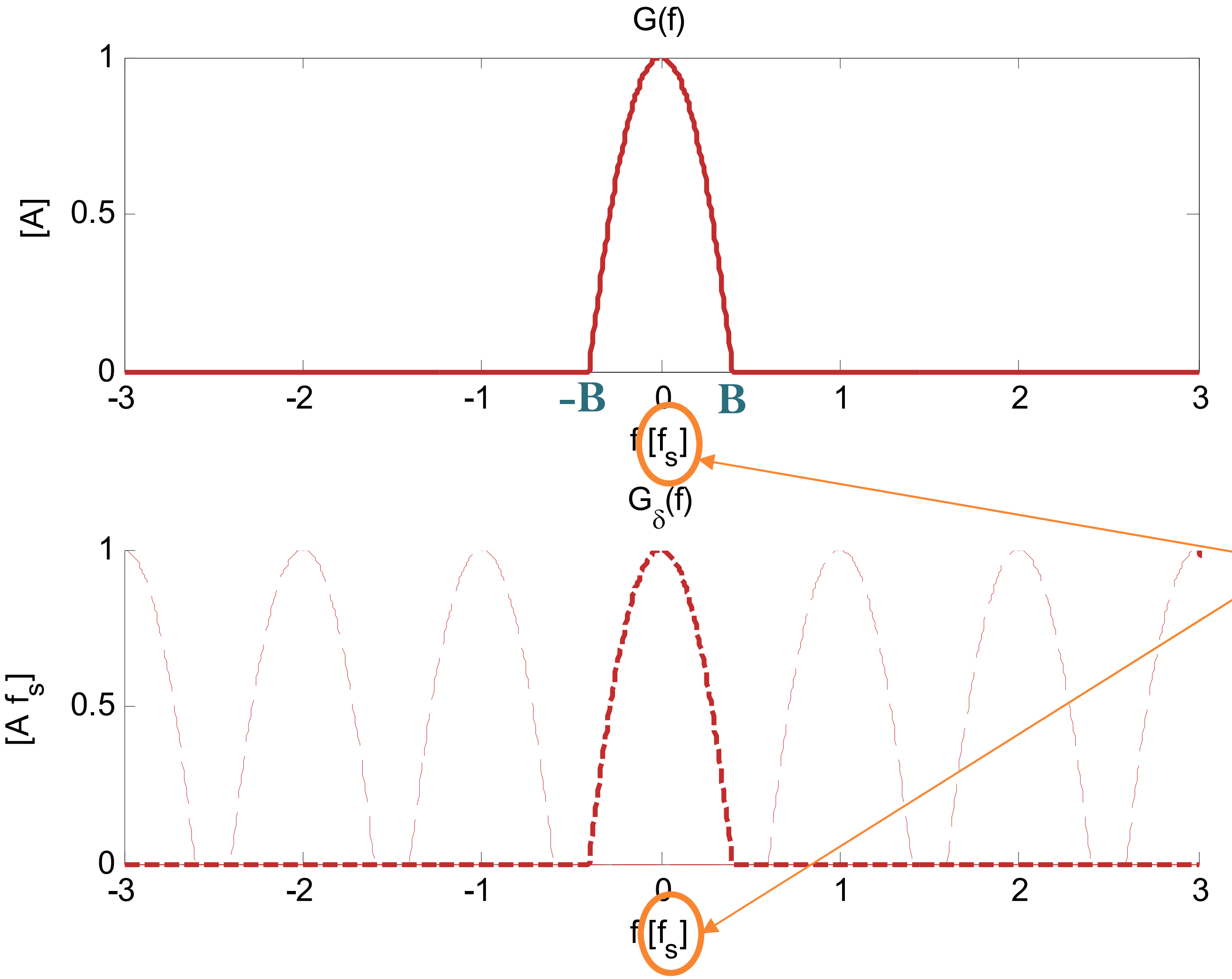


Note the unit used here.

Ideal Sampling: MATLAB Exploration

The Fourier transform of the original signal

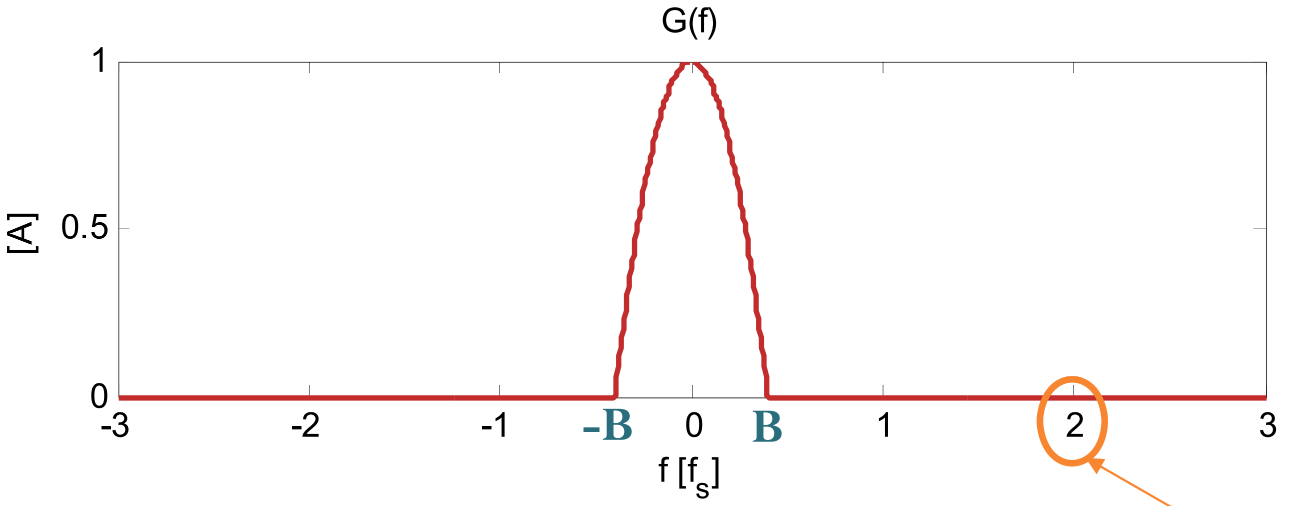
The Fourier transform of the (ideal) sampled signal



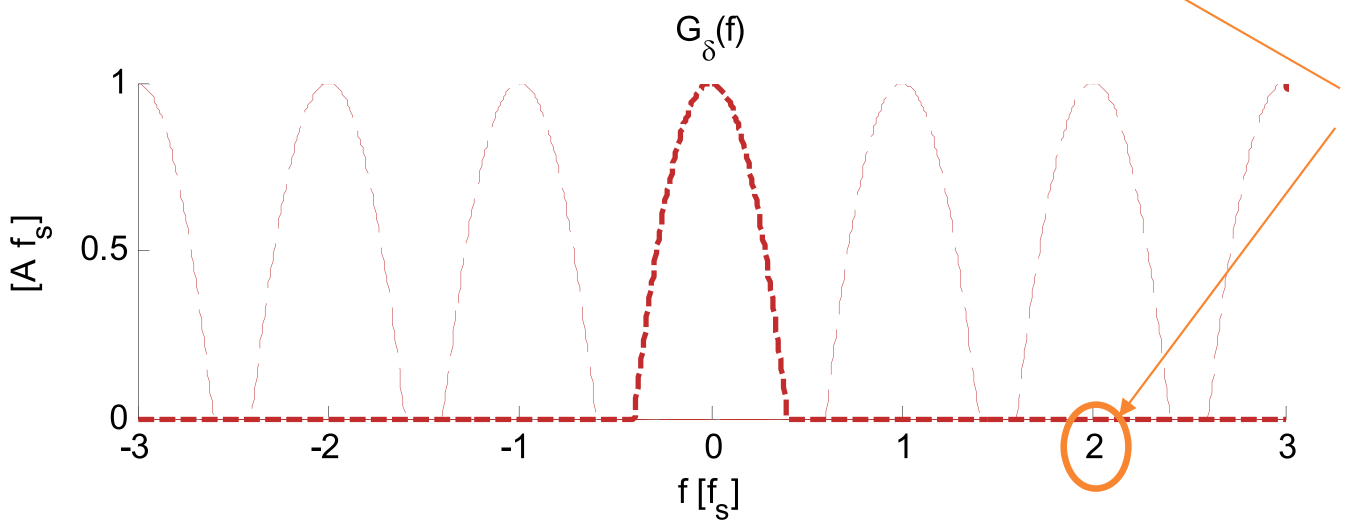
Note that the frequency unit here is f_s.

Ideal Sampling: MATLAB Exploration

The Fourier transform of the original signal



The Fourier transform of the (ideal) sampled signal

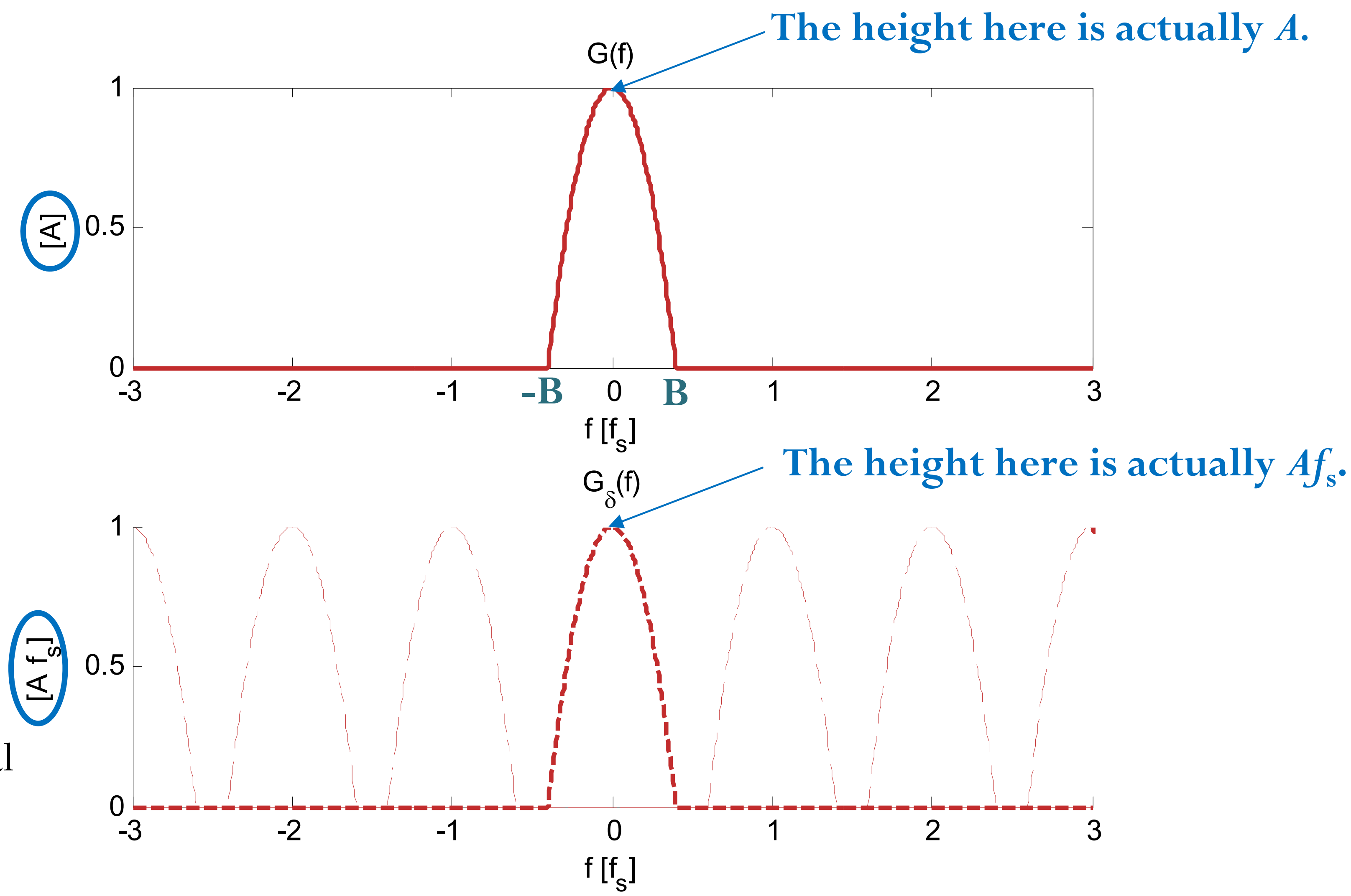


So, it's actually $2f_s$ here.

Ideal Sampling: MATLAB Exploration

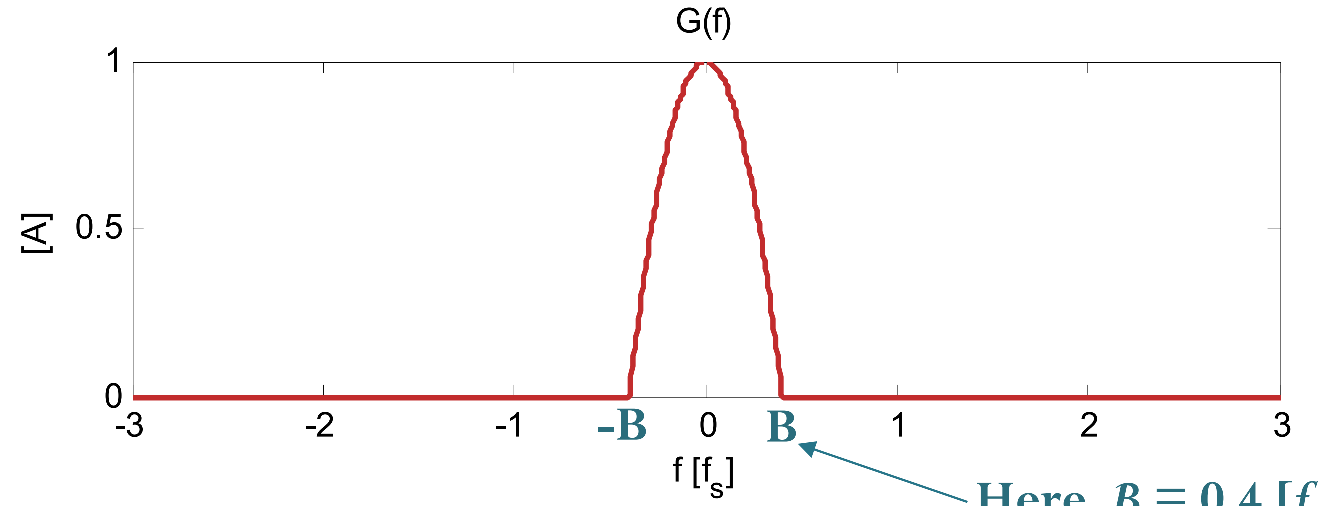
The Fourier transform of the original signal

The Fourier transform of the (ideal) sampled signal



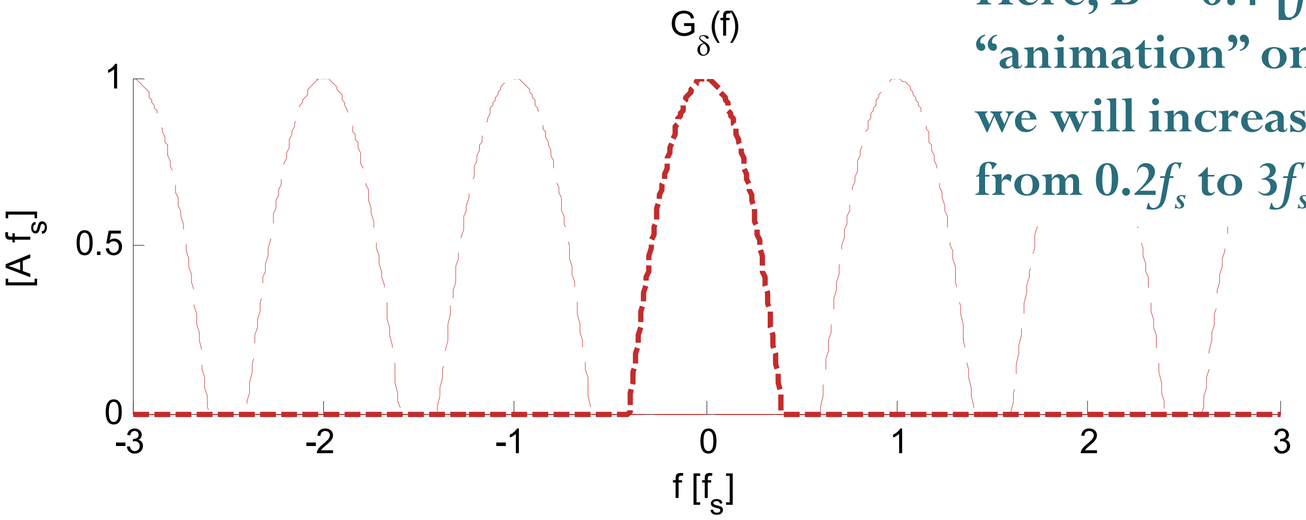
Ideal Sampling: MATLAB Exploration

The Fourier transform of the original signal

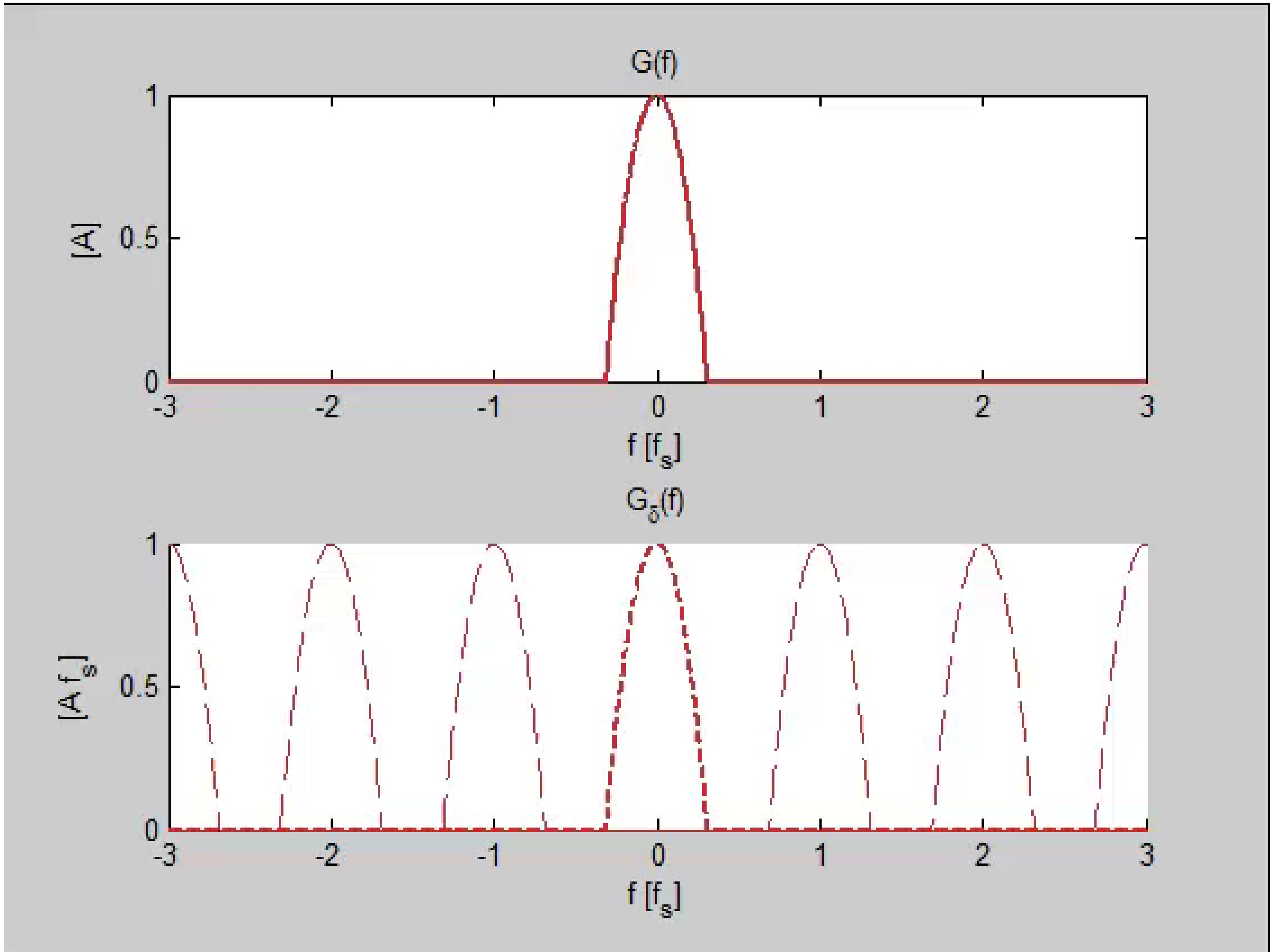


Here, $B = 0.4$ [f_s]. In the "animation" on the next slide, we will increase the value of B from $0.2f_s$ to $3f_s$.

The Fourier transform of the (ideal) sampled signal



Ideal Sampling: MATLAB Exploration



When the value of *B* is too large, the shifted replicas of *G*(*f*) overlap.

[Gdelta_demo1.m]

Ideal Sampling: MATLAB Exploration

The Fourier transform of the original signal

The Fourier transform of the (ideal) sampled signal

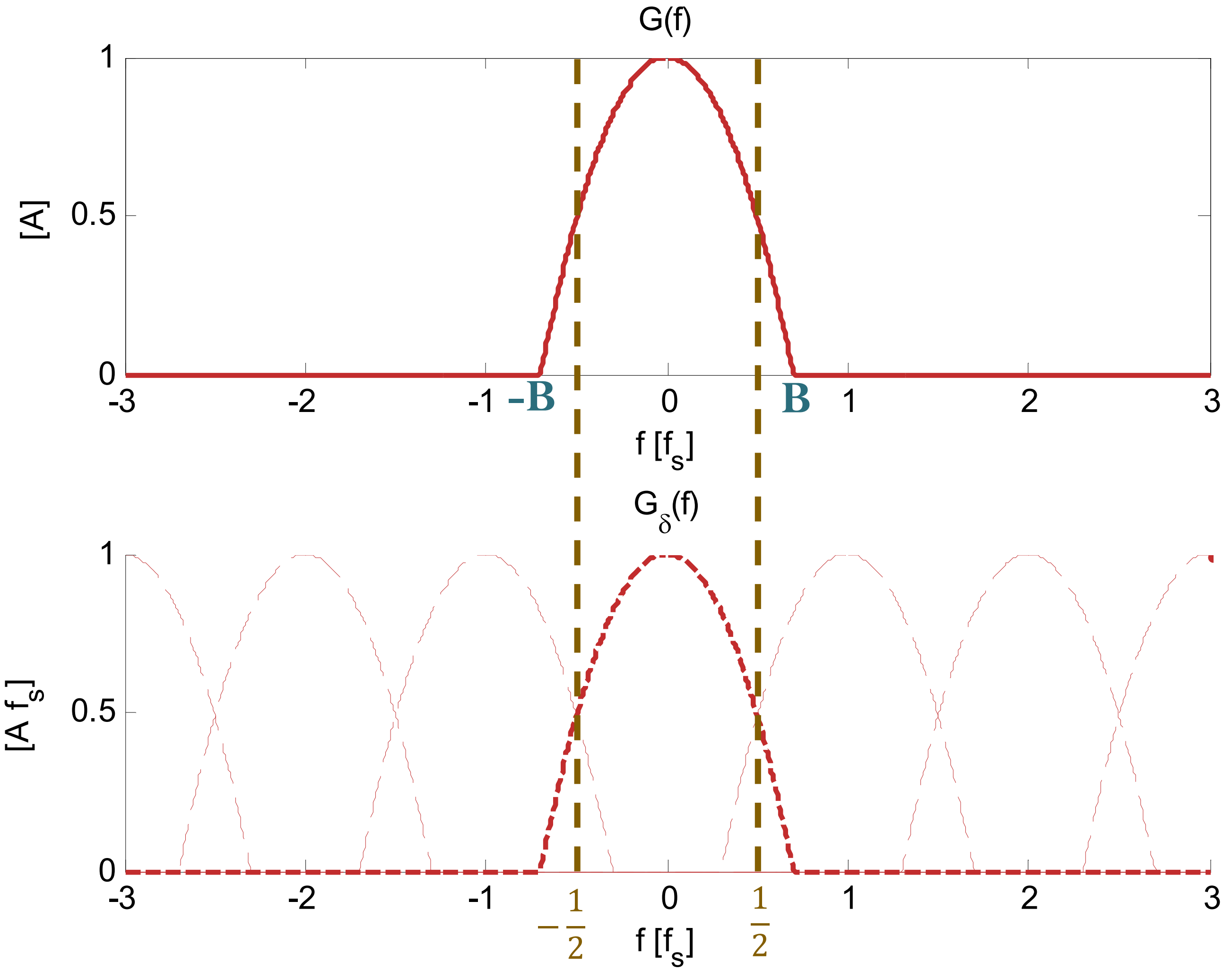


Figure 48

When $B > 0.5$ $[f_s]$, overlapping happens in the frequency domain.

This spectral overlapping of the signal is commonly referred to as "aliasing".

Ideal Sampling: MATLAB Exploration

The Fourier transform of the original signal

The Fourier transform of the (ideal) sampled signal

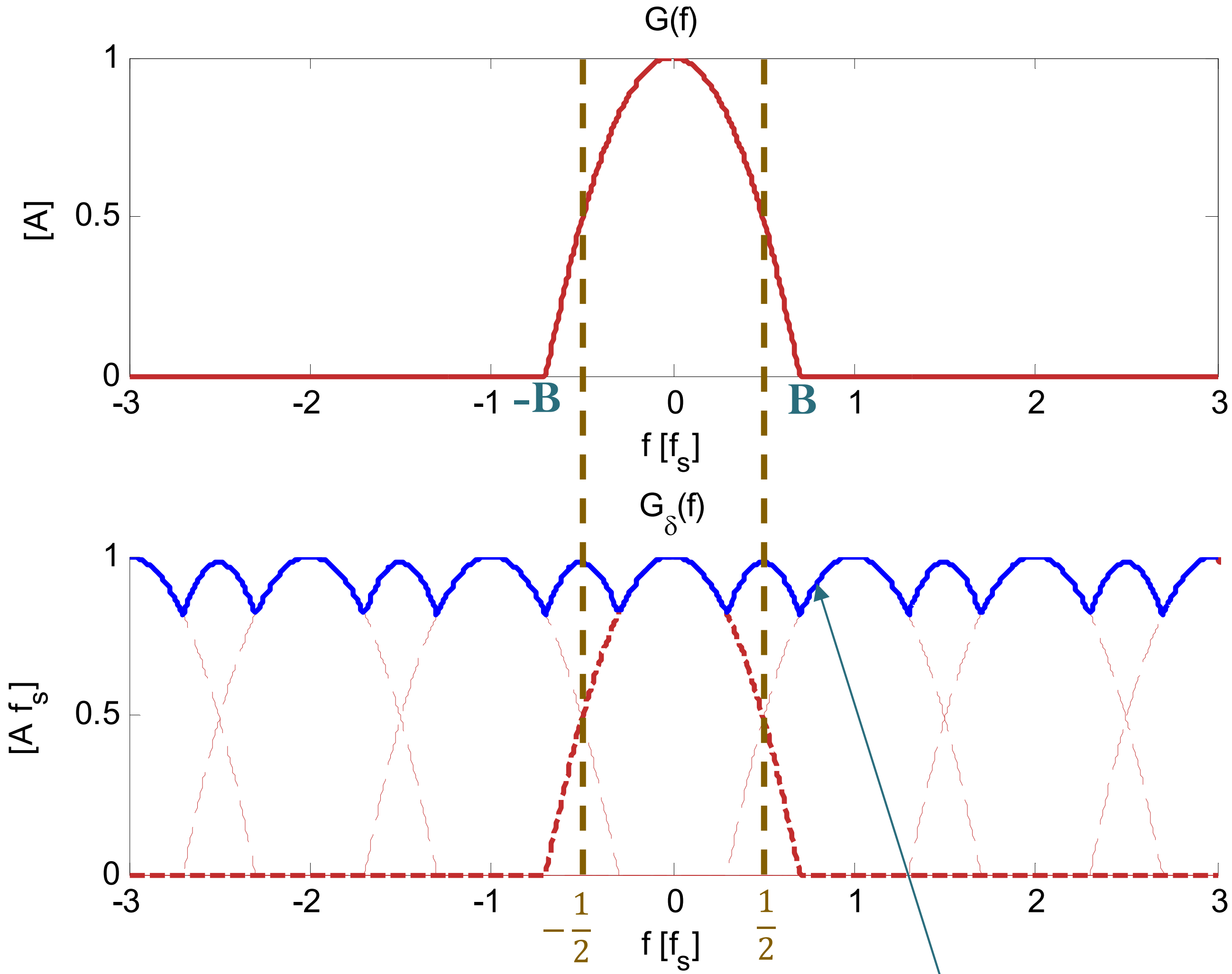
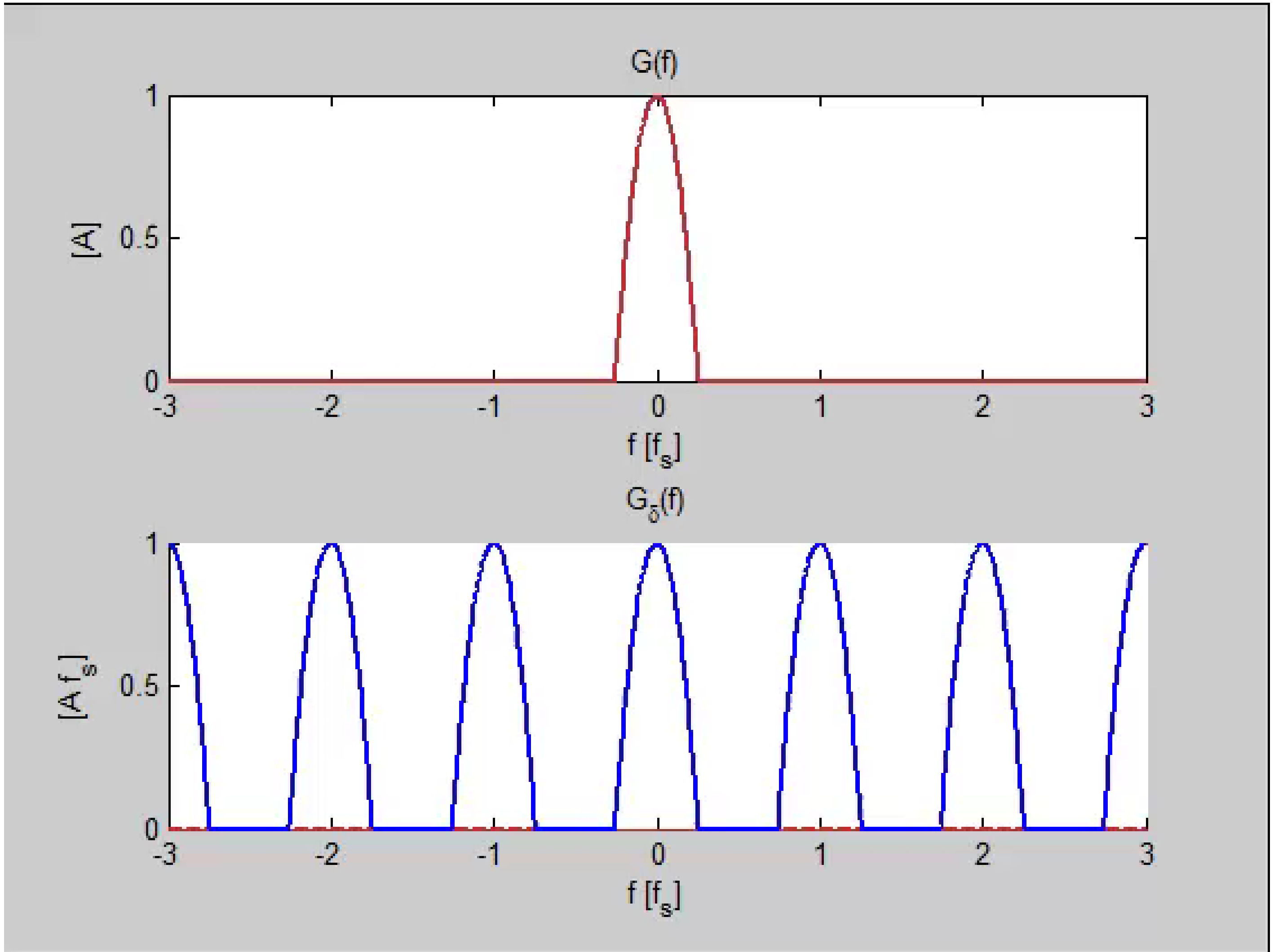


Figure 48

To find $G_\delta(f)$, don't forget to add the replicas

Very different from *G*(*f*).

Ideal Sampling: MATLAB Exploration



[Gdelta_demo2.m]

Ideal Sampling: MATLAB Exploration

The Fourier transform of the original signal

The Fourier transform of the (ideal) sampled signal

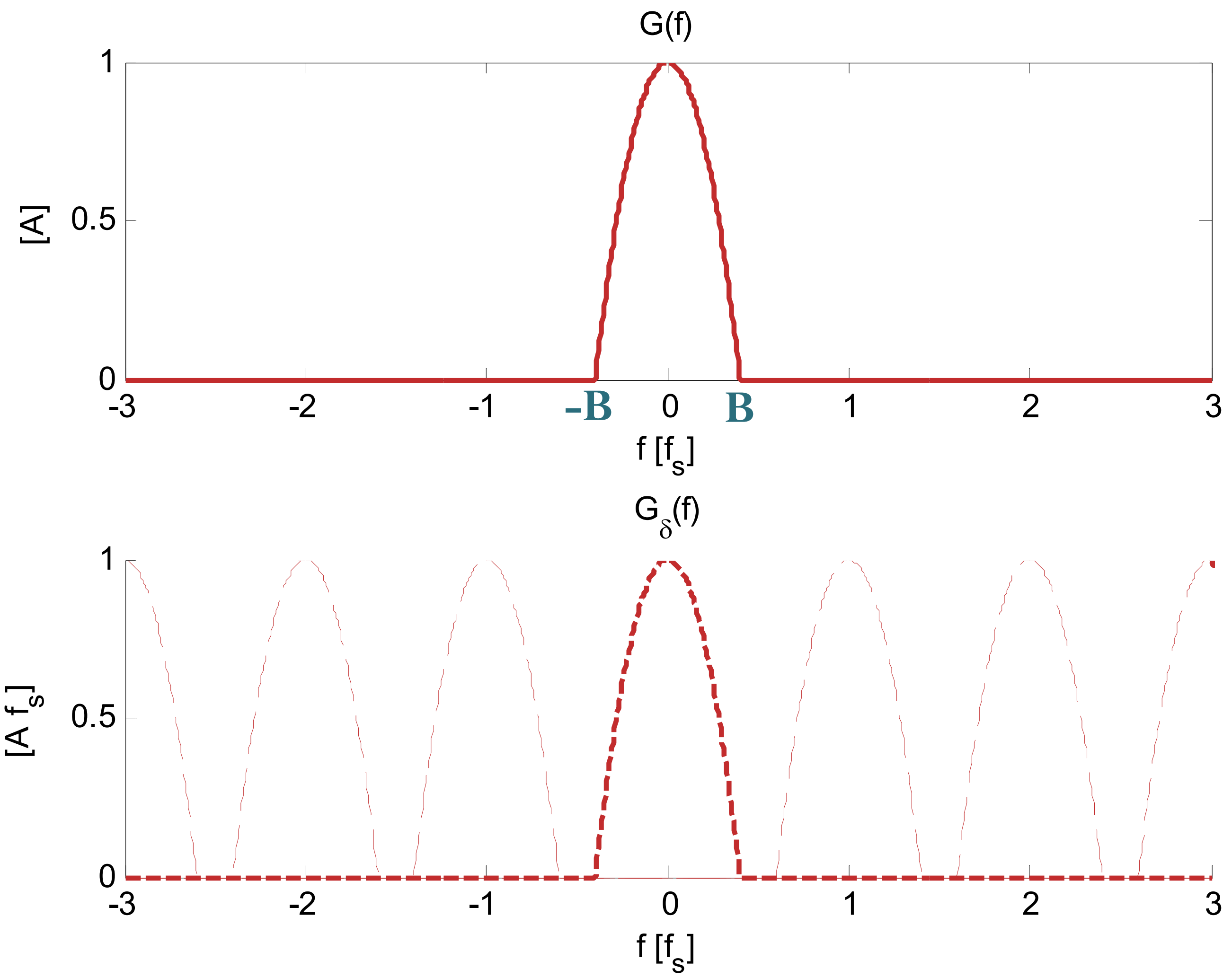


Figure 47

When $B < 0.5$ [f_s], the replicas do not overlap and hence we do not need to spend extra effort to find their sum.

Ideal Sampling: MATLAB Exploration

The Fourier transform of the original signal

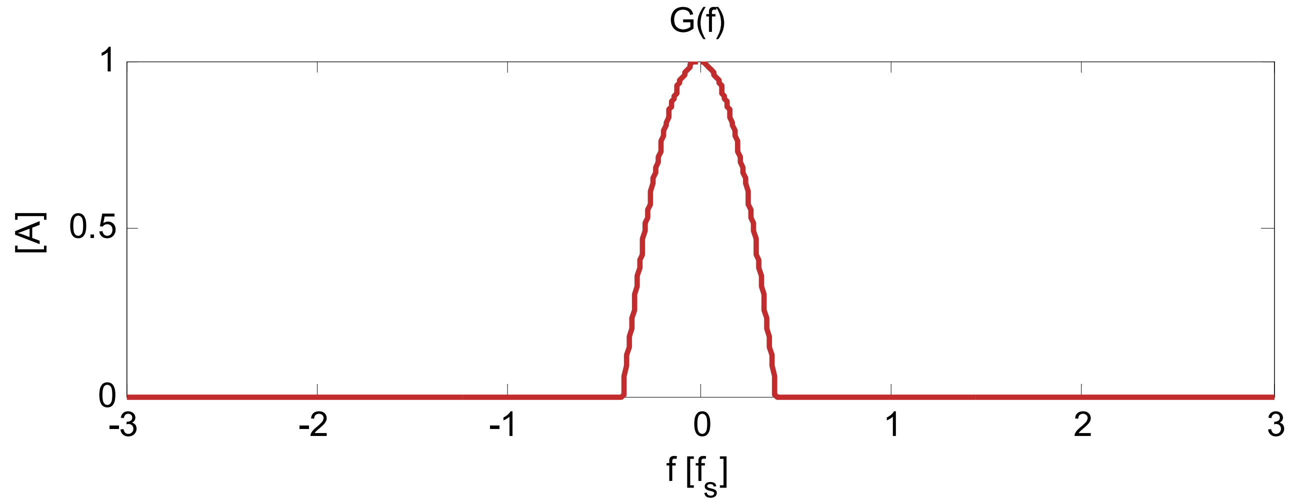
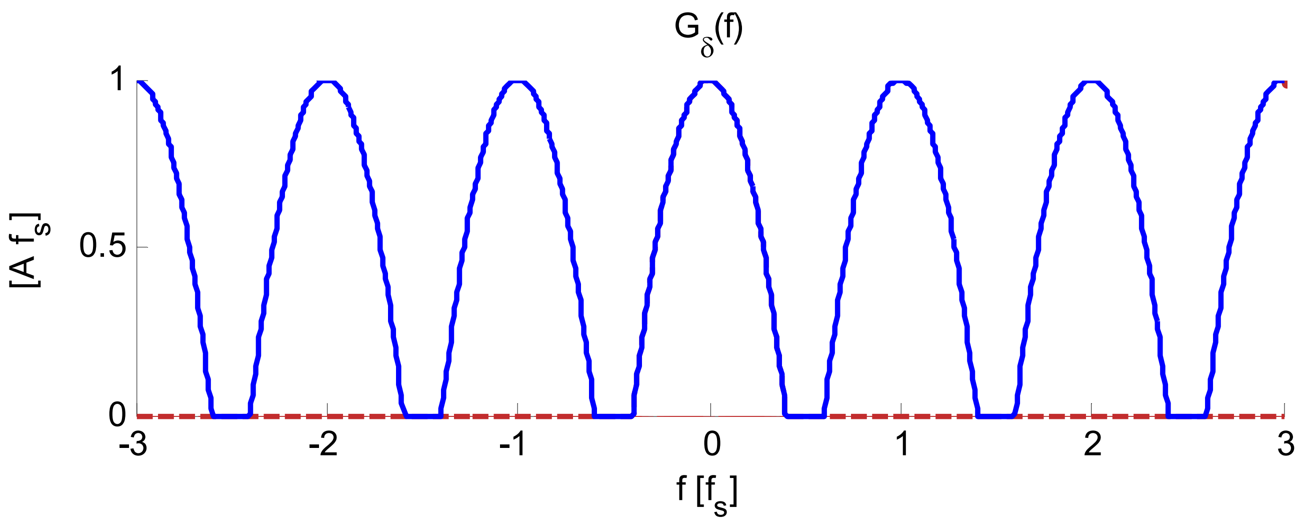


Figure 47

When $B < 0.5$ $[f_s]$, the replicas do not overlap and hence we do not need to spend extra effort to find their sum.

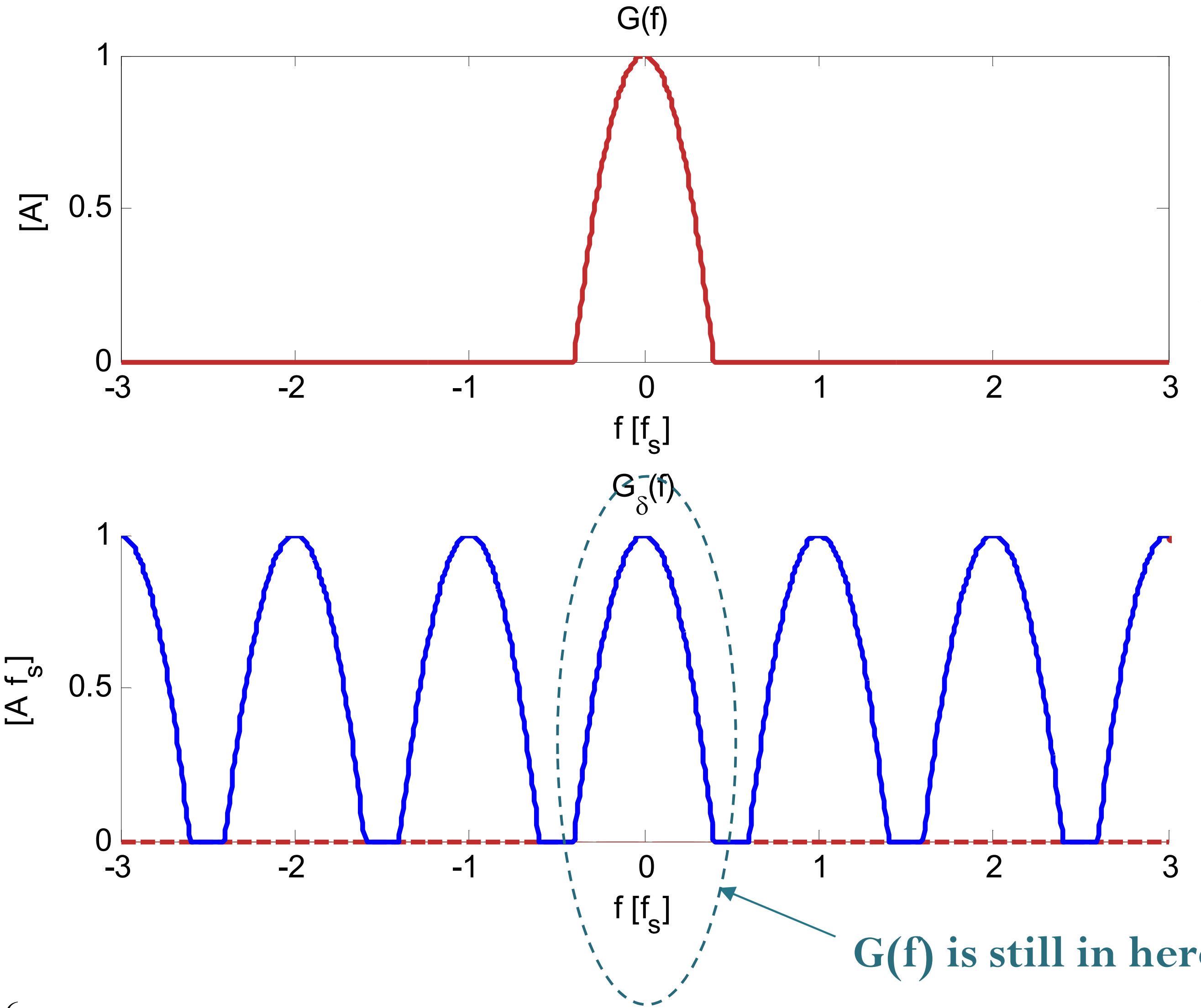
The Fourier transform of the (ideal) sampled signal



Ideal Sampling: MATLAB Exploration

The Fourier transform of the original signal

The Fourier transform of the (ideal) sampled signal



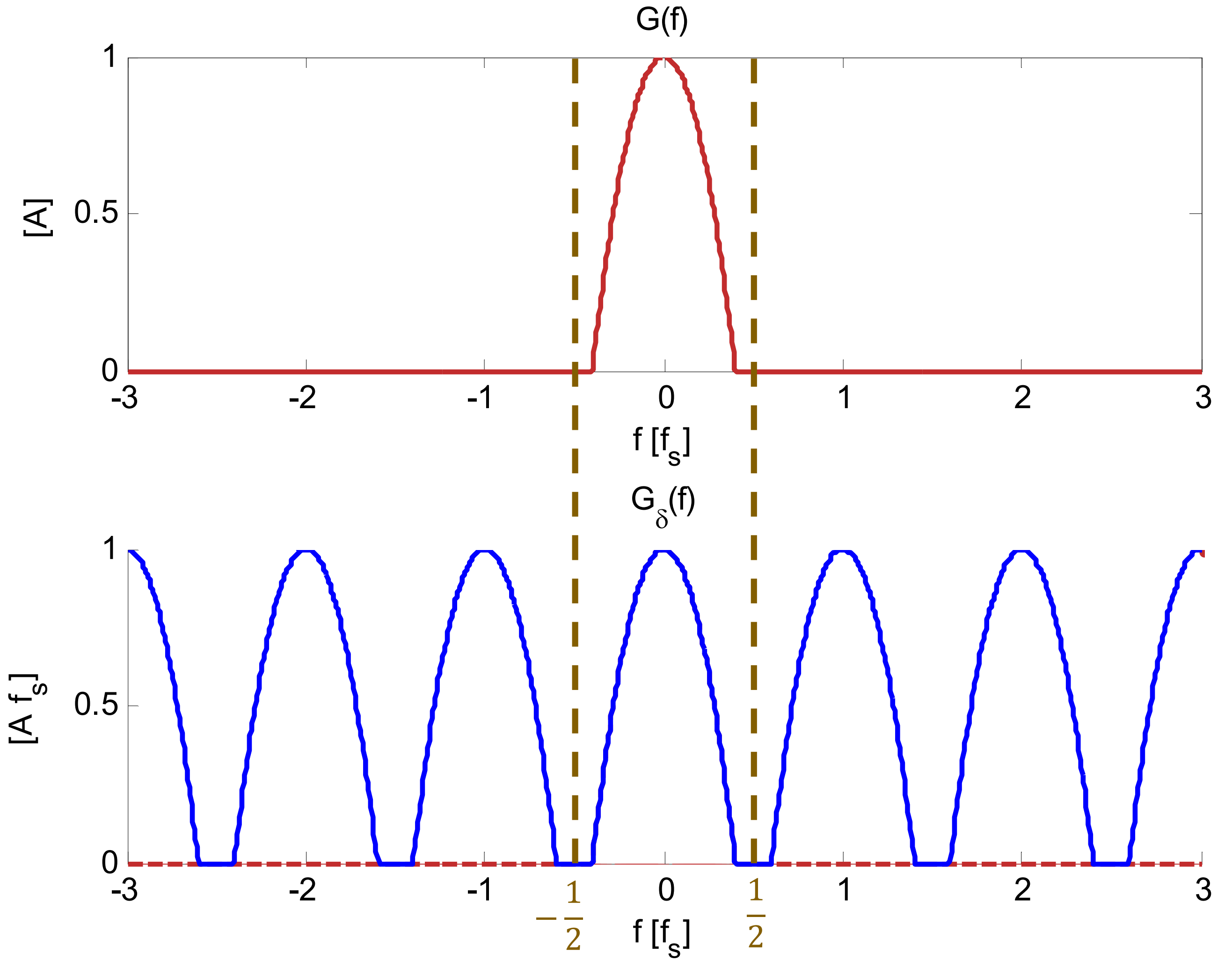
When $B < 0.5$ [f_s], the replicas do not overlap and hence we do not need to spend extra effort to find their sum.

G(f) is still in here.

Ideal Sampling: Periodicity

The Fourier transform of the original signal

The Fourier transform of the (ideal) sampled signal

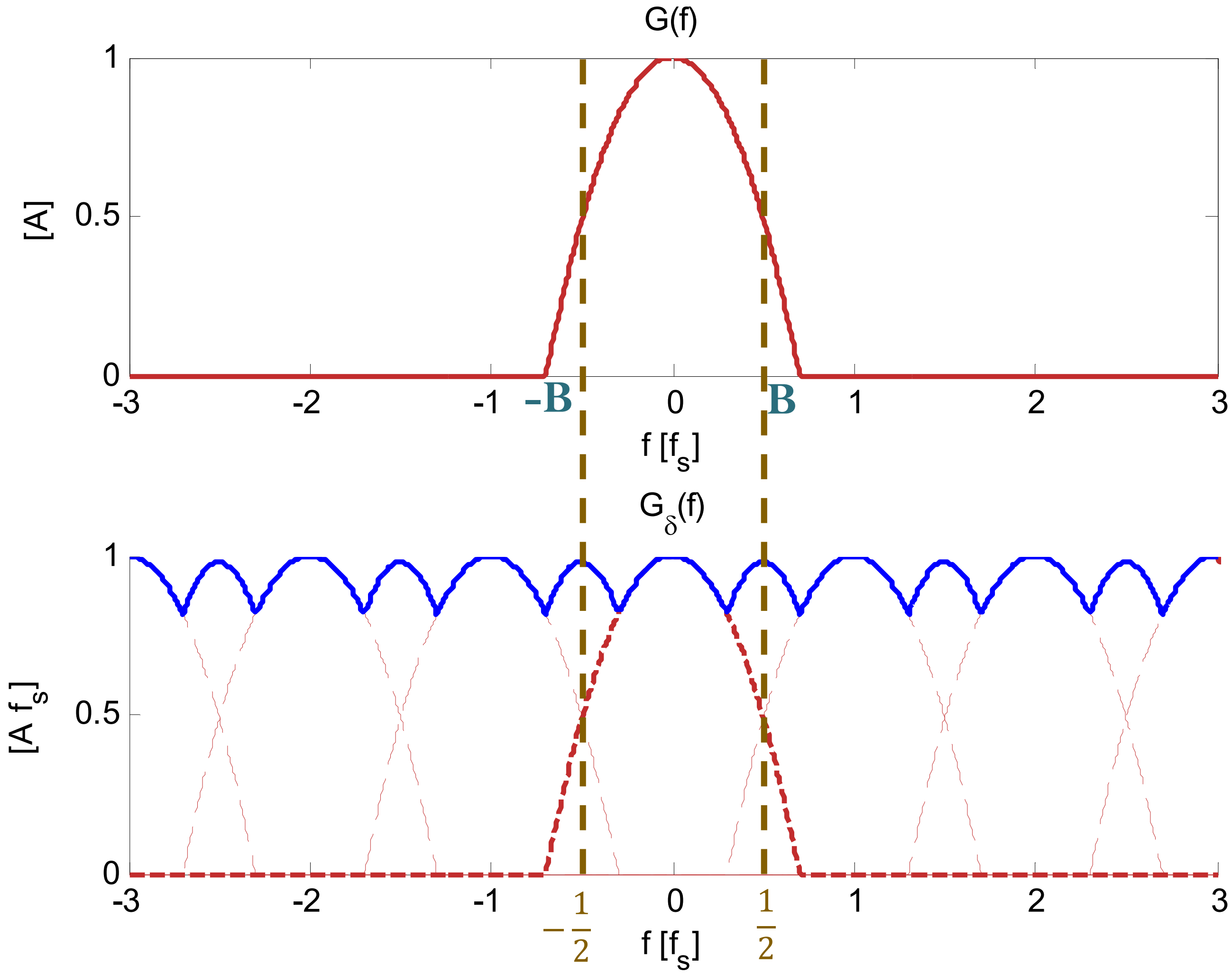


Note that $G_\delta(f)$ is "periodic" in the frequency domain with period f_s. Therefore, it is sufficient to look only at f between $\pm\frac{f_s}{2}$.

Ideal Sampling: Periodicity

The Fourier transform of the original signal

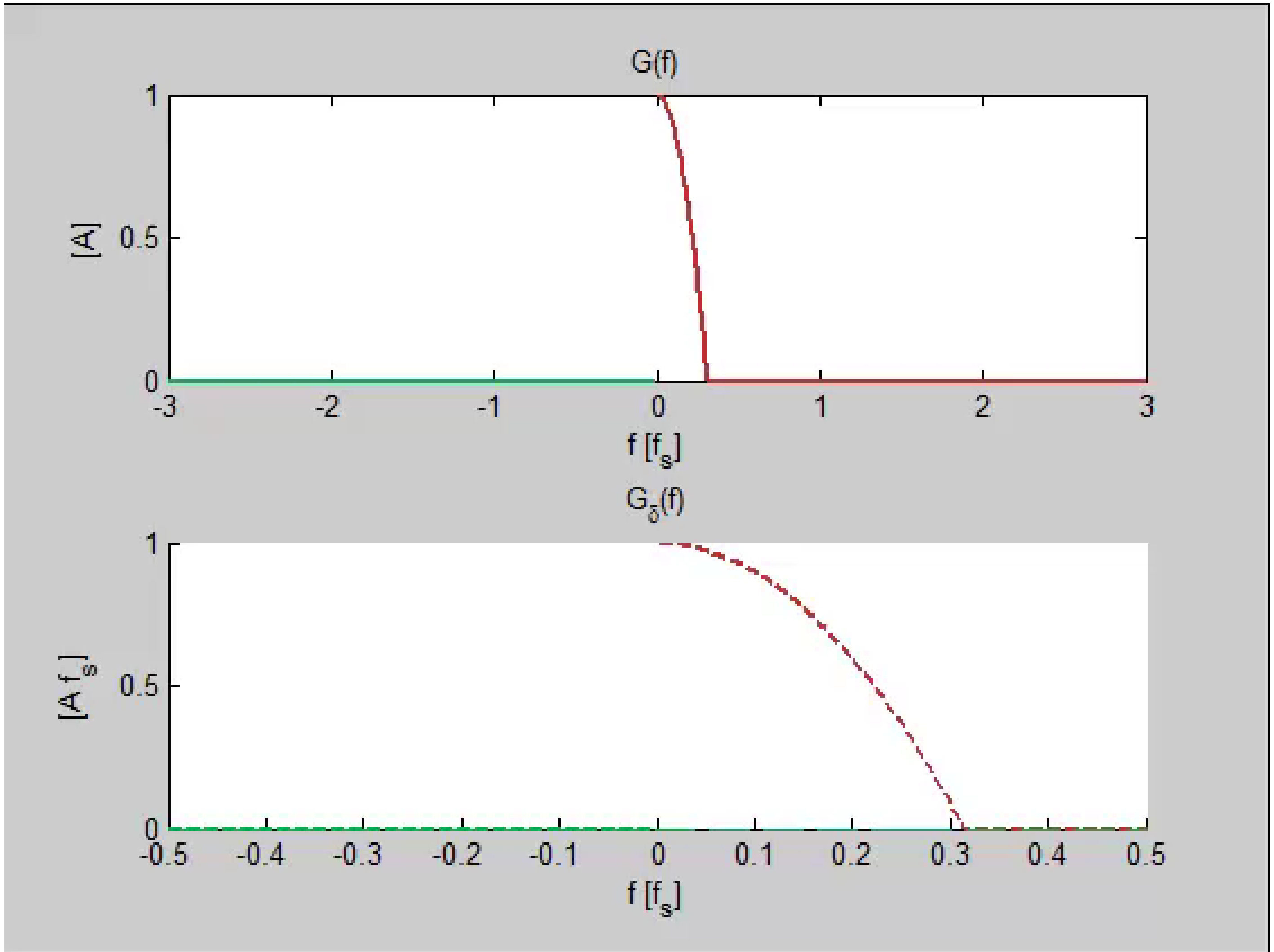
The Fourier transform of the (ideal) sampled signal



The periodicity holds regardless of whether the replicas overlap in the frequency domain.

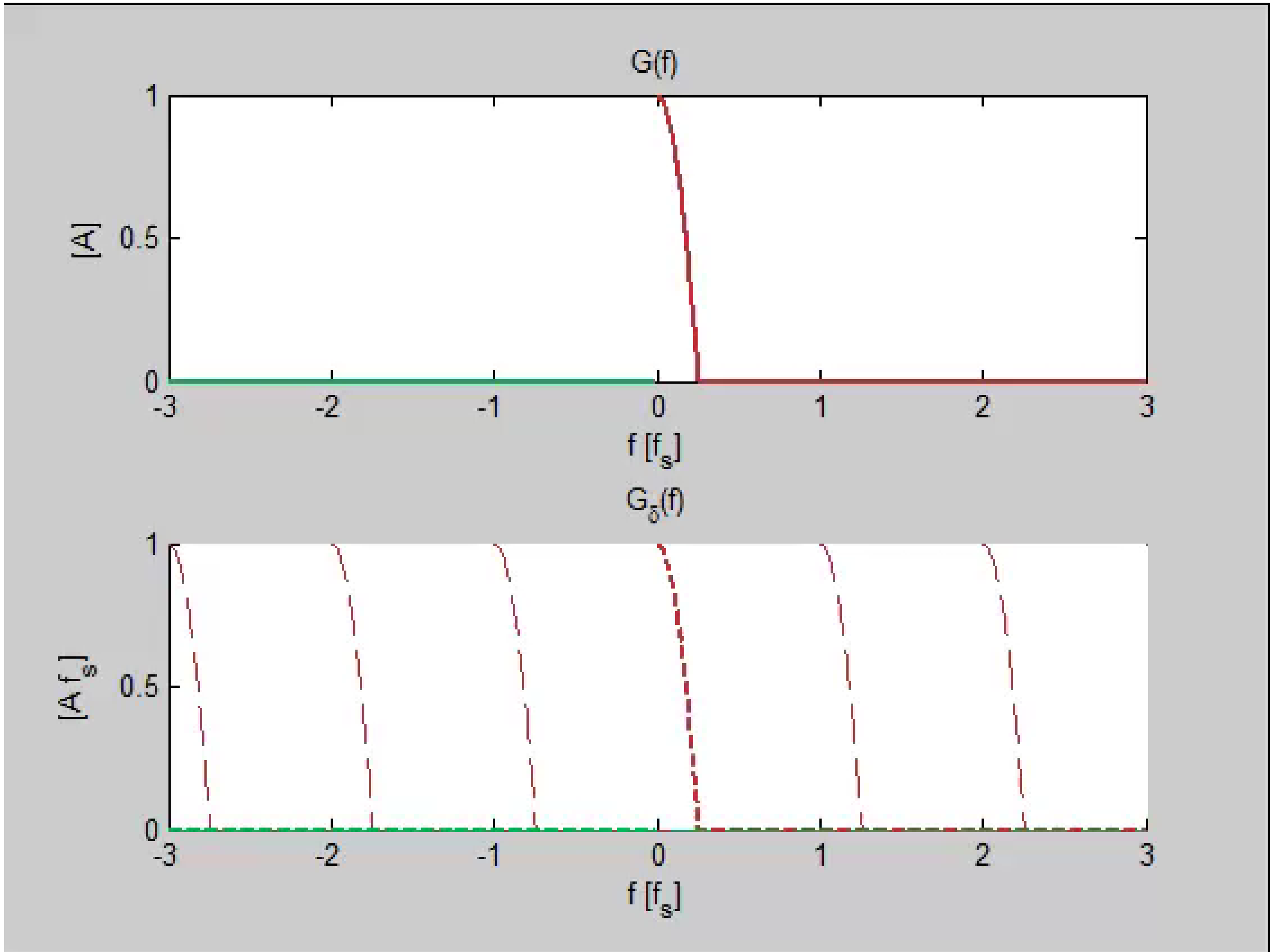
Note that $G_\delta(f)$ is "periodic" in the frequency domain with period f_s. Therefore, it is sufficient to look only at f between $\pm\frac{f_s}{2}$.

Ideal Sampling: Tunneling



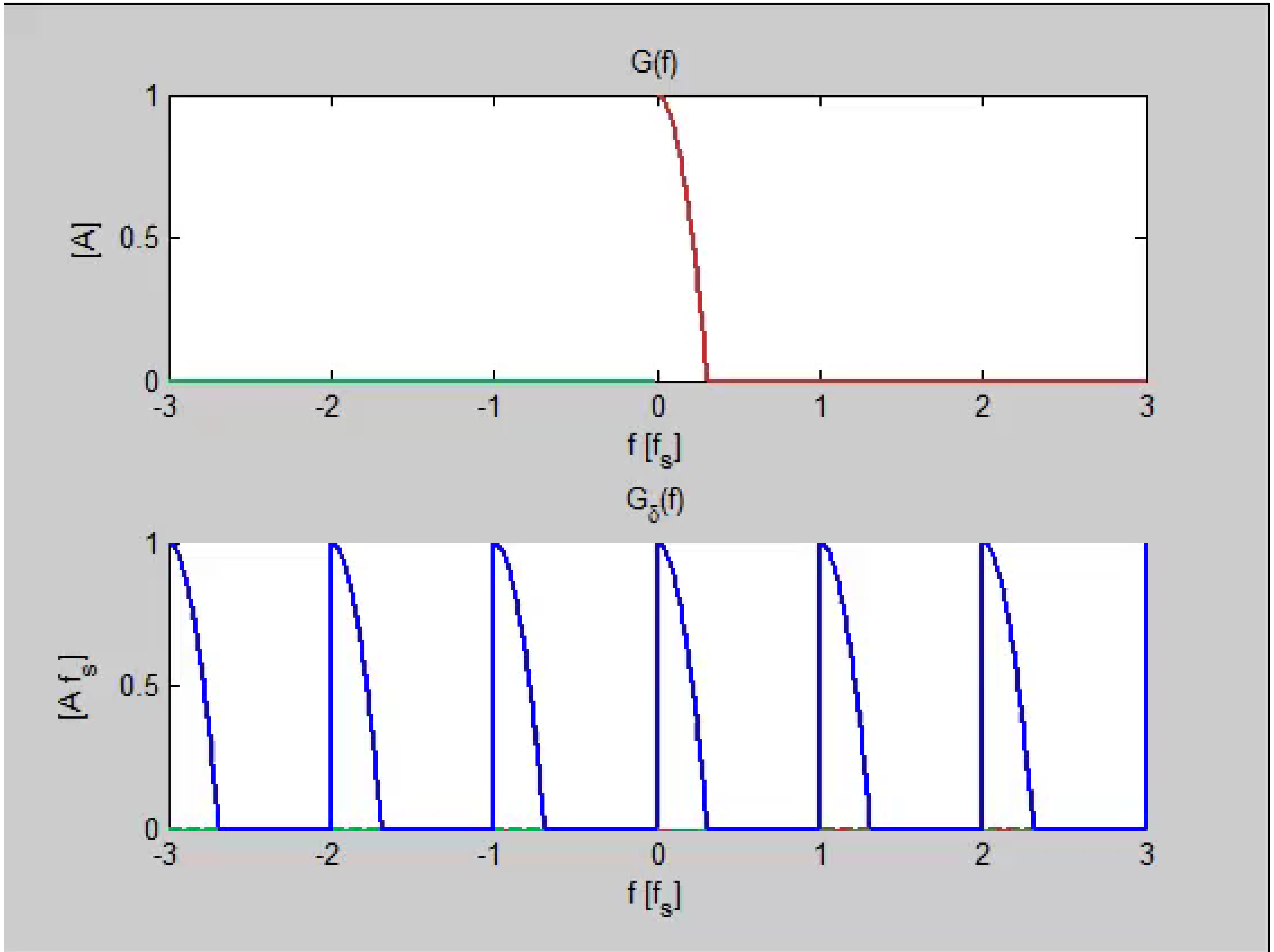
[Gdelta_demo5.m]

Ideal Sampling: Tunneling



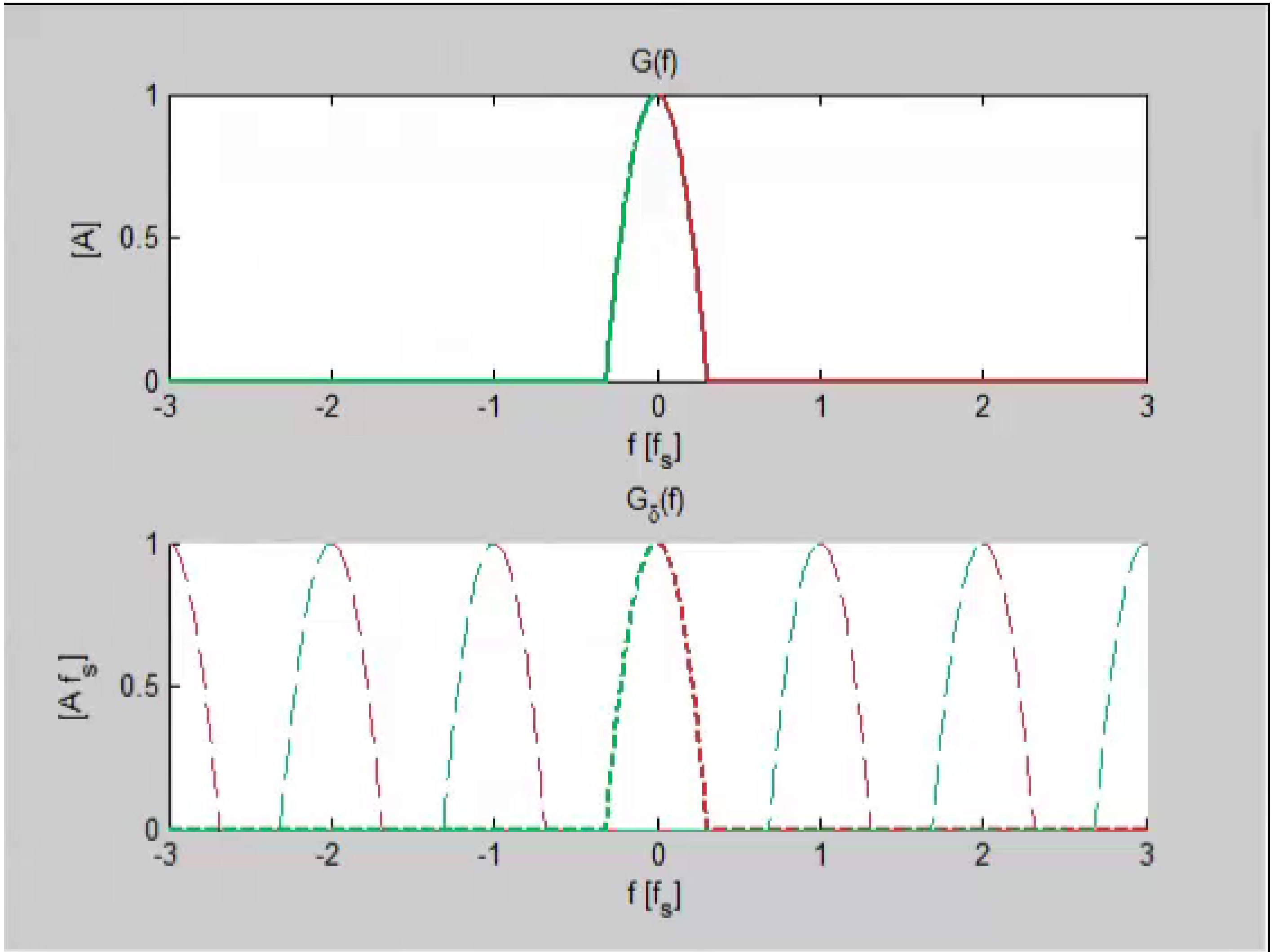
[Gdelta_demo7.m]

Ideal Sampling: Tunneling



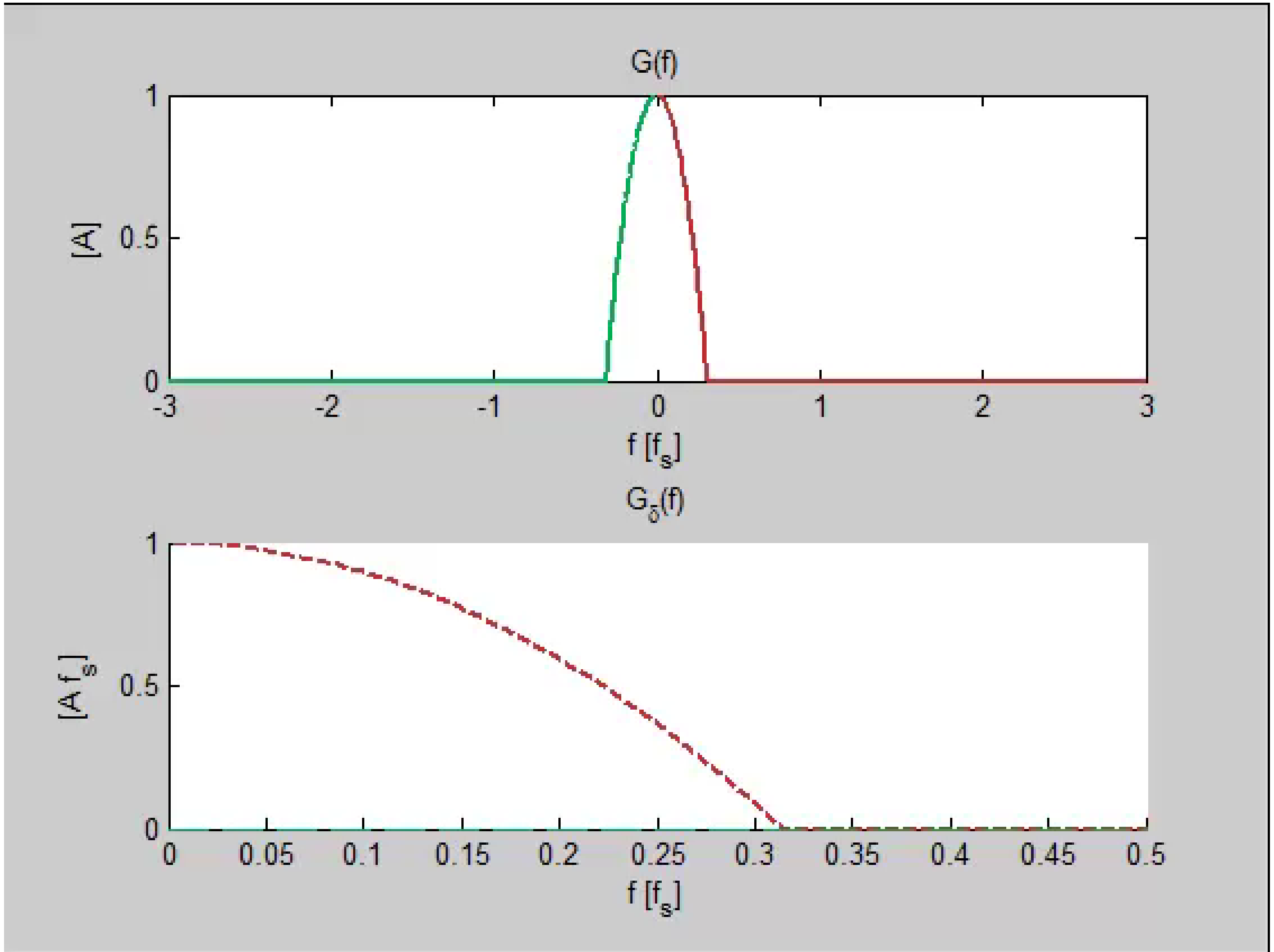
[Gdelta_demo8.m]

Ideal Sampling: Folding



[Gdelta_demo10.m]

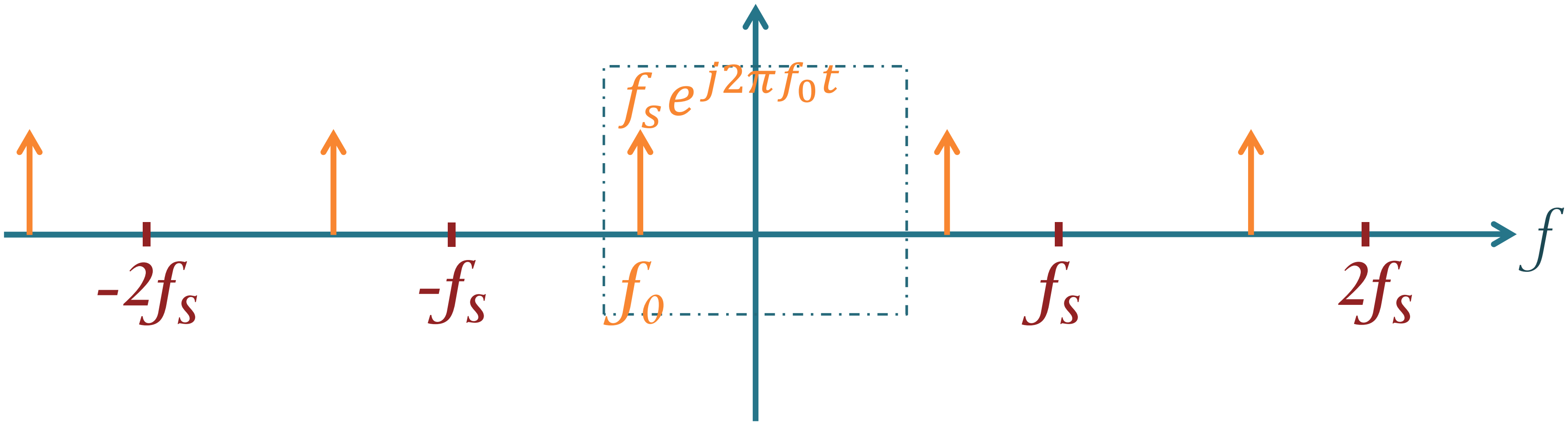
Ideal Sampling: Folding (a revisit)



[Gdelta_demo9.m]

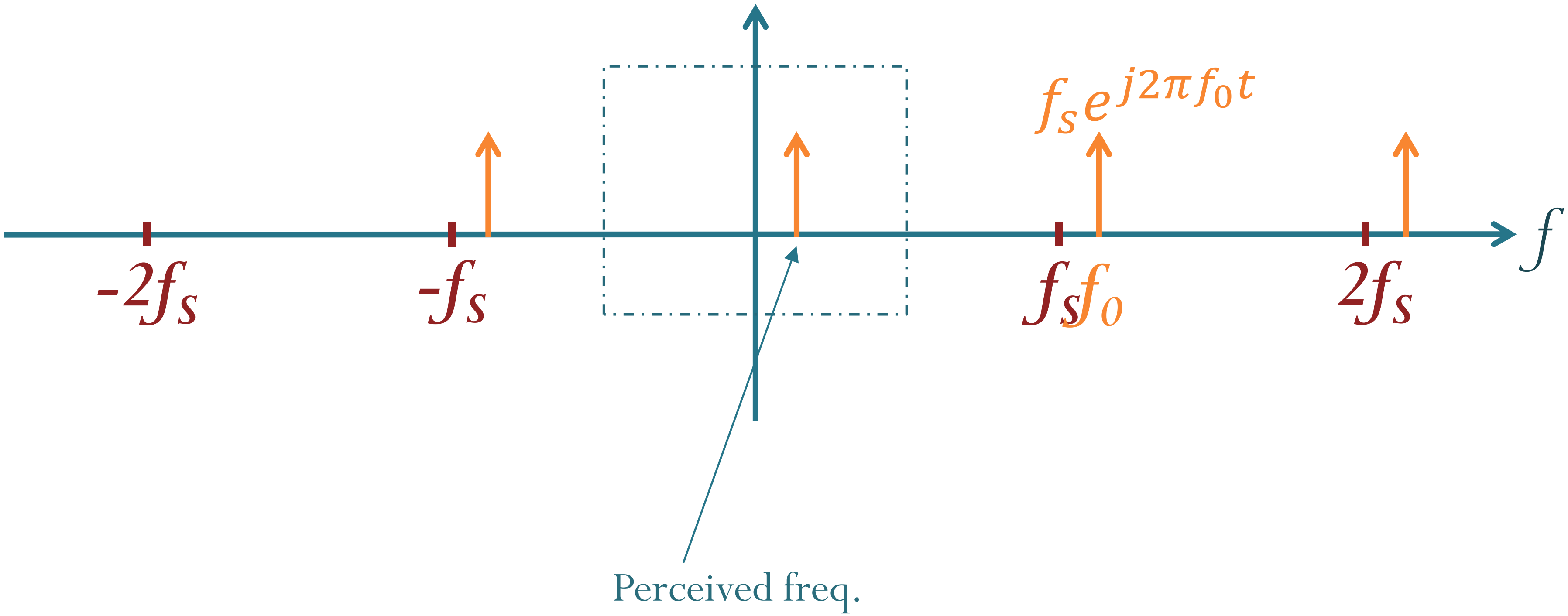
Complex exponential

Sample a complex-exponential signal $e^{j2\pi f_0 t}$ with sampling rate f_s



Let's increase f_0

Complex exponential



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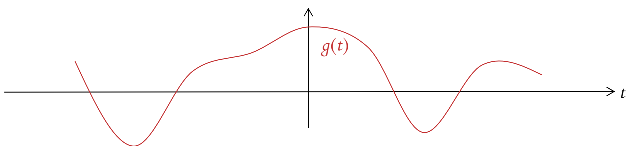
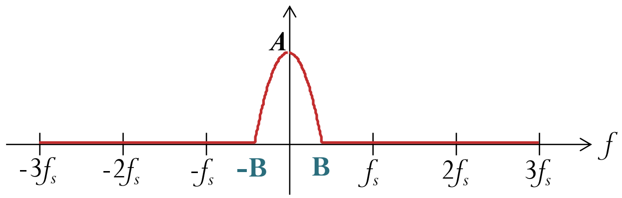
6.3 Reconstruction

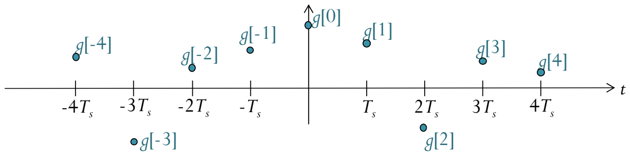
Sampling and Reconstruction

Time Domain **Frequency Domain**

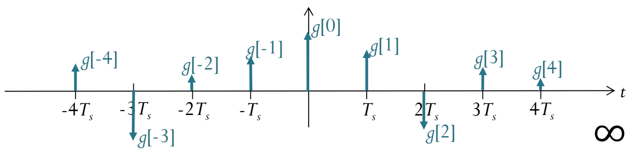


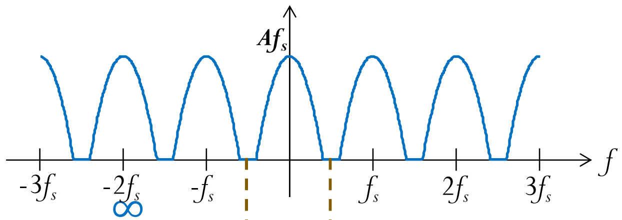
$$g(t) \rightleftharpoons^{\mathcal{F}} G(f)$$



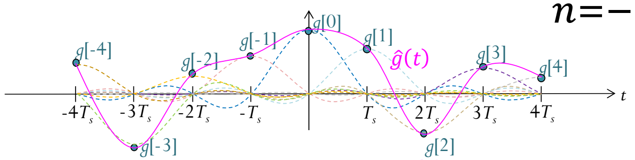


$$g[n] = g(nT_s)$$

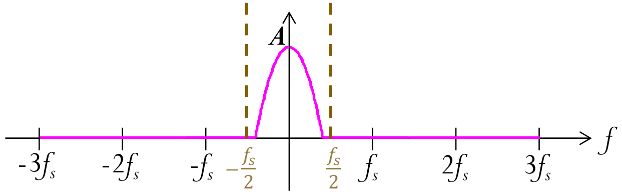




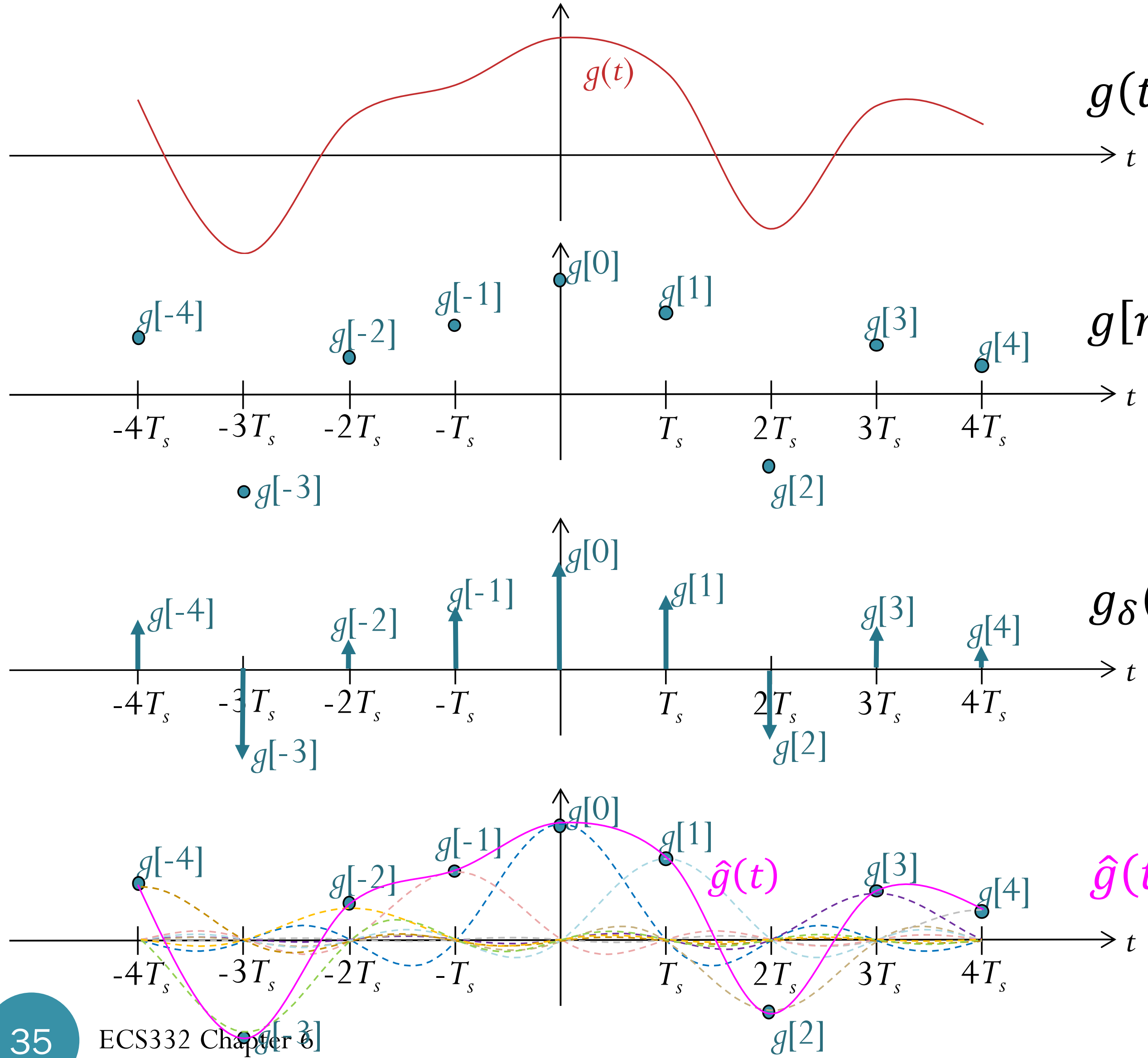
$$g_\delta(t) = \sum_{n=-\infty}^{\infty} g[n]\,\delta(t - nT_s) \rightleftharpoons^{\mathcal{F}} G_\delta(f) = \sum_{k=-\infty}^{\infty} f_s G(f - kf_s)$$



$$\hat{g}(t) = \sum_{n=-\infty}^{\infty} g[n]\,\mathrm{sinc}\big(\pi f_s (t - nT_s)\big) \rightleftharpoons^{\mathcal{F}} \hat{G}(f)$$



Sampling and Reconstruction



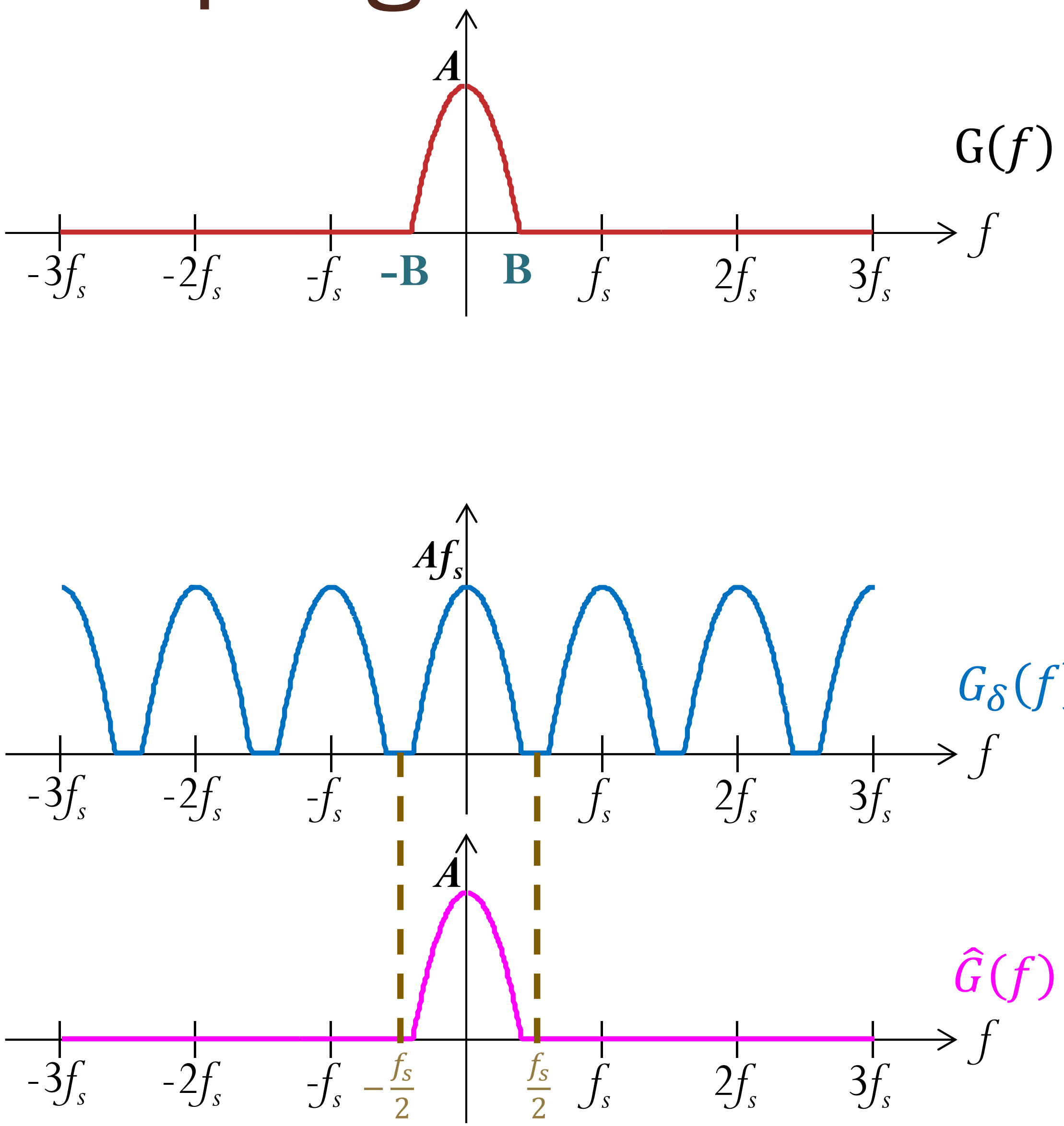
$$g(t)$$

$$g[n] = g(nT_s)$$

$$g_\delta(t) = \sum_{n=-\infty}^{\infty} g[n]\,\delta(t - nT_s)$$

$$\hat{g}(t) = \sum_{n=-\infty}^{\infty} g[n]\,\mathrm{sinc}\big(\pi f_s (t - nT_s)\big)$$

Sampling and Reconstruction

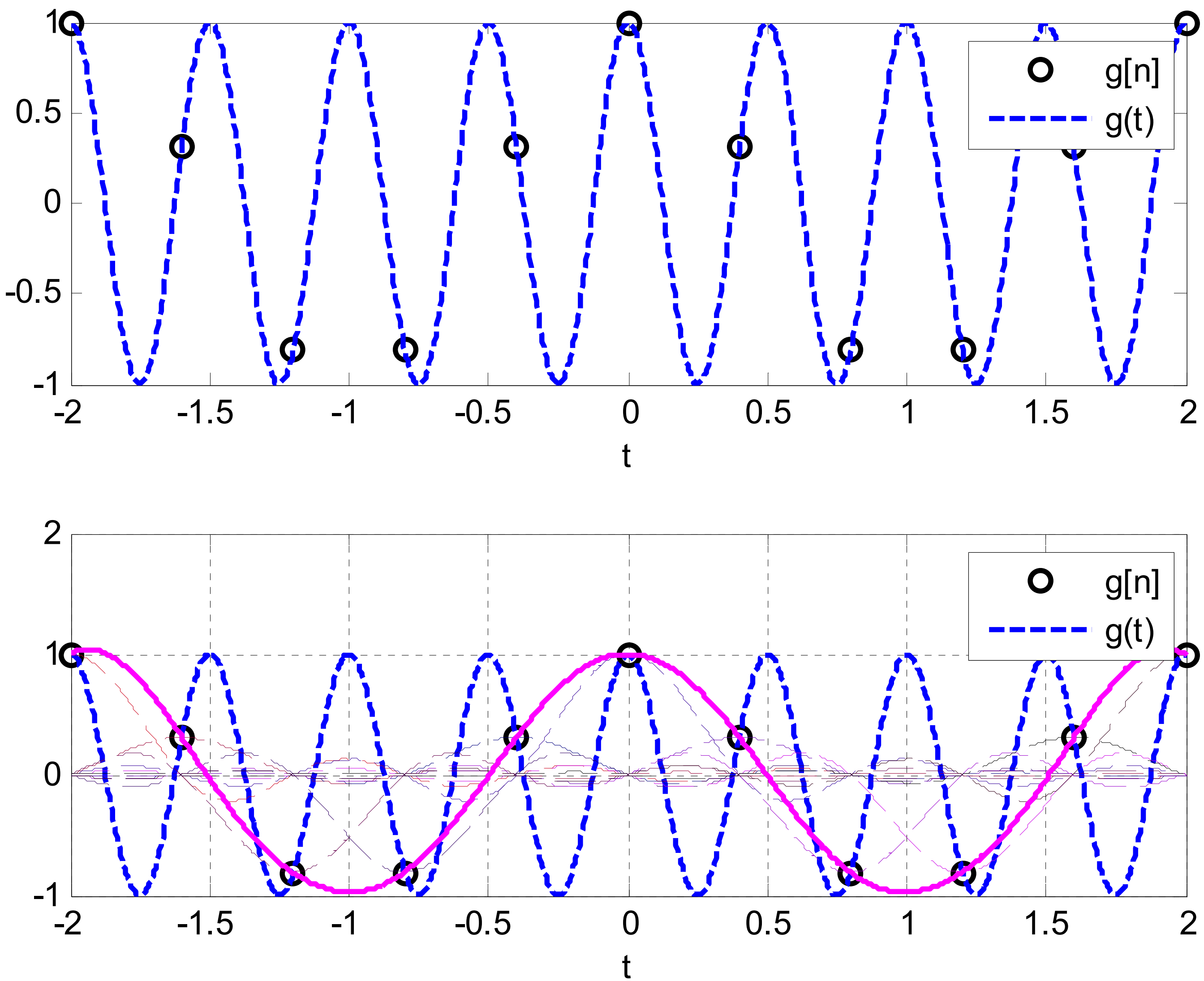


$$G_\delta(f) = \sum_{k=-\infty}^{\infty} f_s G(f - k f_s)$$

[Example 6.31]

Reconstruction of $\cos(2\pi(2)t)$

$T_s = 0.4$



[Upper plot in Figure 50.]

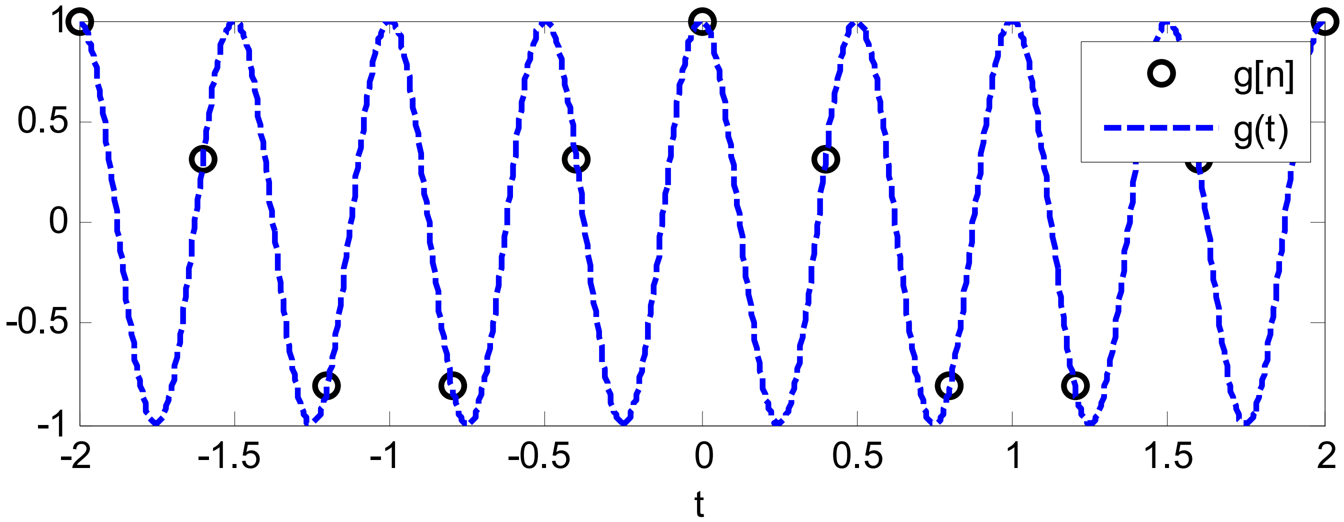
B = 2 Hz.

[Example 6.31]

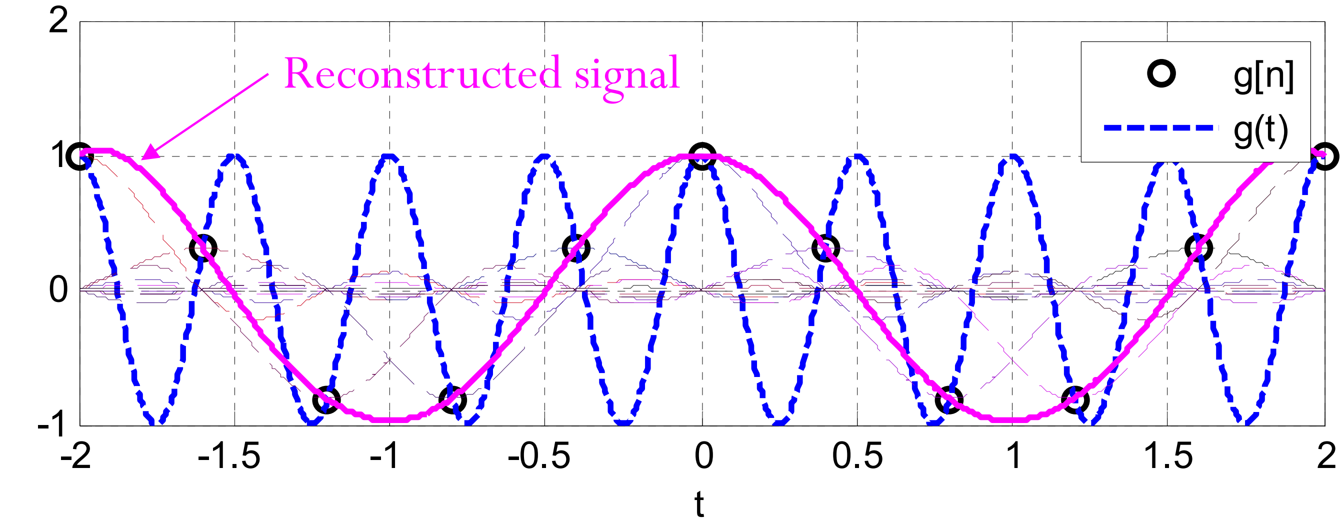
Reconstruction of $\cos(2\pi(2)t)$

$T_s = 0.4$

$f_s = 1/0.4$
$= 2.5$ [Sa/s]

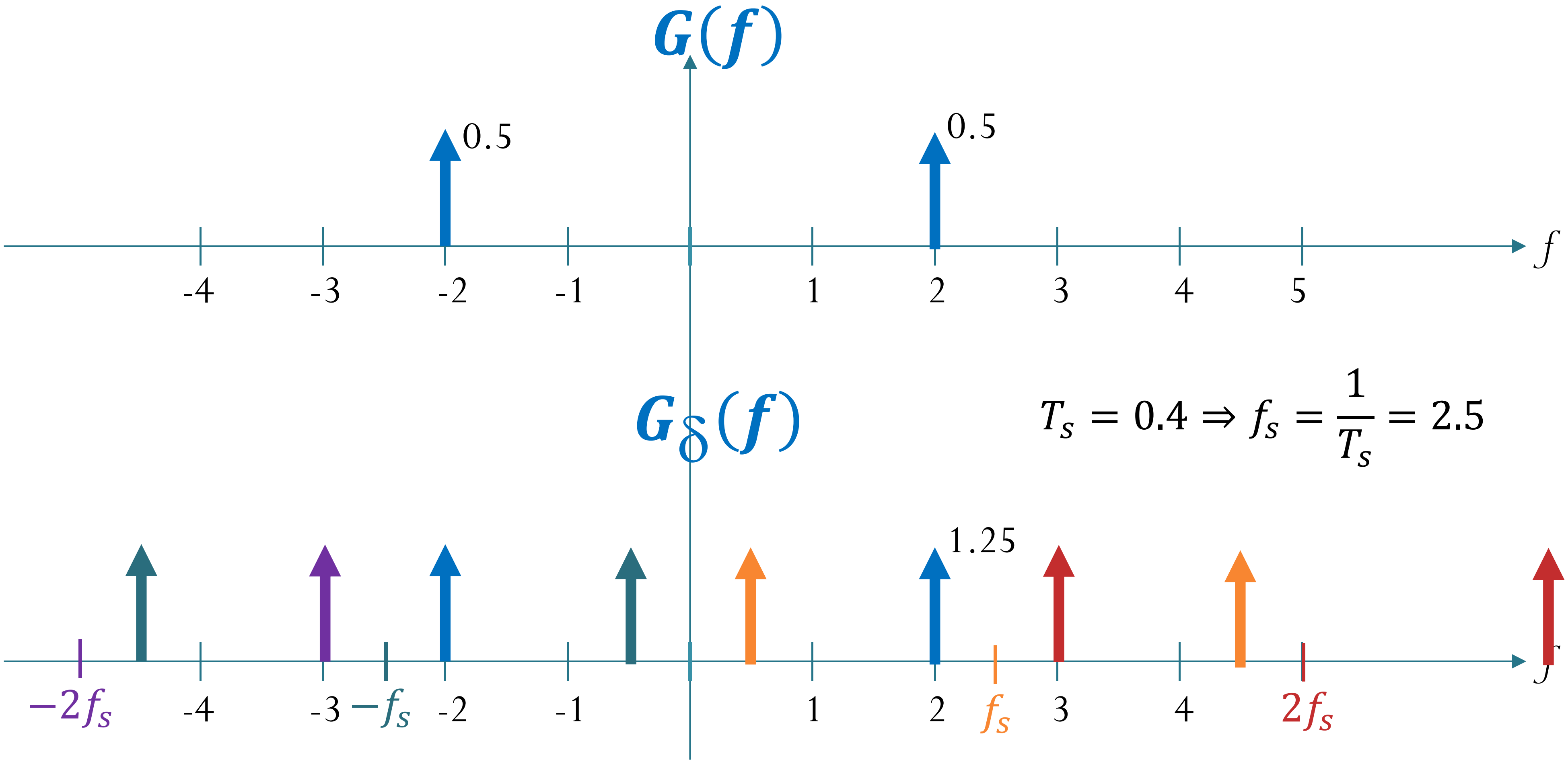


[Upper plot in Figure 50.]



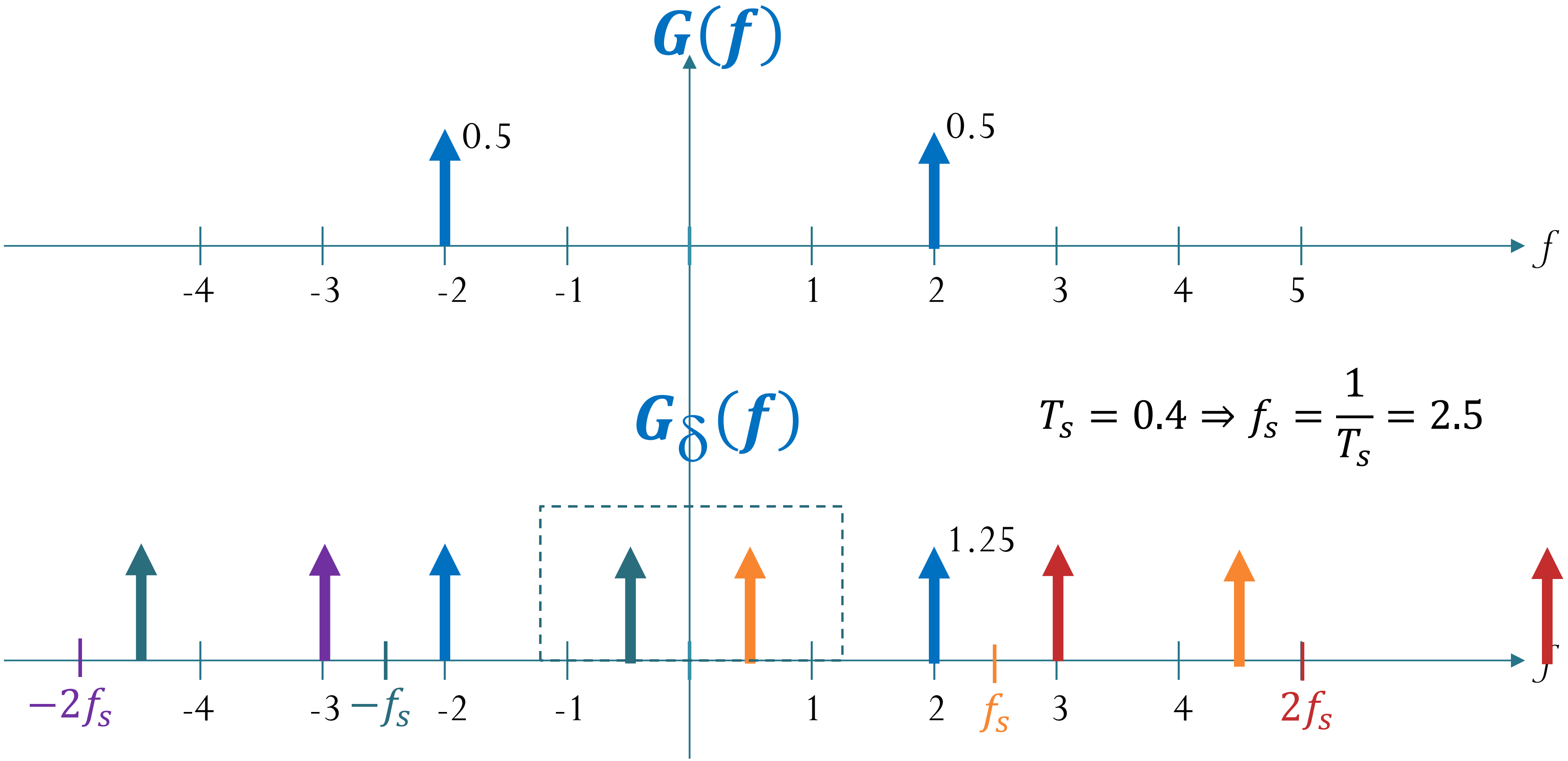
$f_s < 2B \Rightarrow$ the reconstructed signal is different from the original signal.

$G_\delta(f)$ when $g(t) = \cos(2\pi(2)t)$



$T_s = 0.4 \Rightarrow f_s = \frac{1}{T_s} = 2.5$

Reconstruction of $g(t) = \cos(2\pi(2)t)$



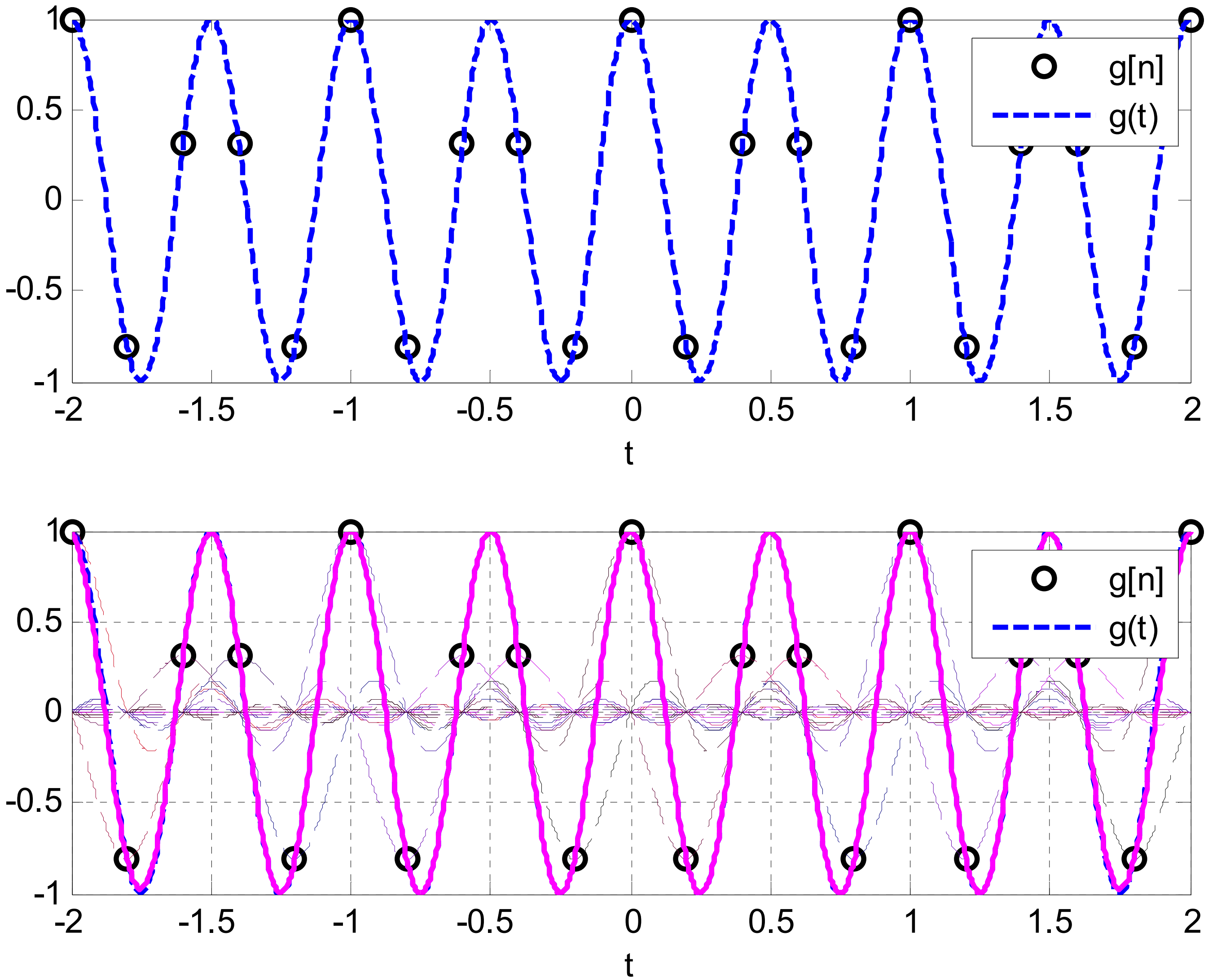
$$T_s = 0.4 \Rightarrow f_s = \frac{1}{T_s} = 2.5$$

$$\hat{g}(t) = \cos(2\pi(0.5)t)$$

Reconstruction of $\cos(2\pi(2)t)$

$T_s = 0.2$

$f_s = \frac{1}{0.2} = 5$ [Sa/s]



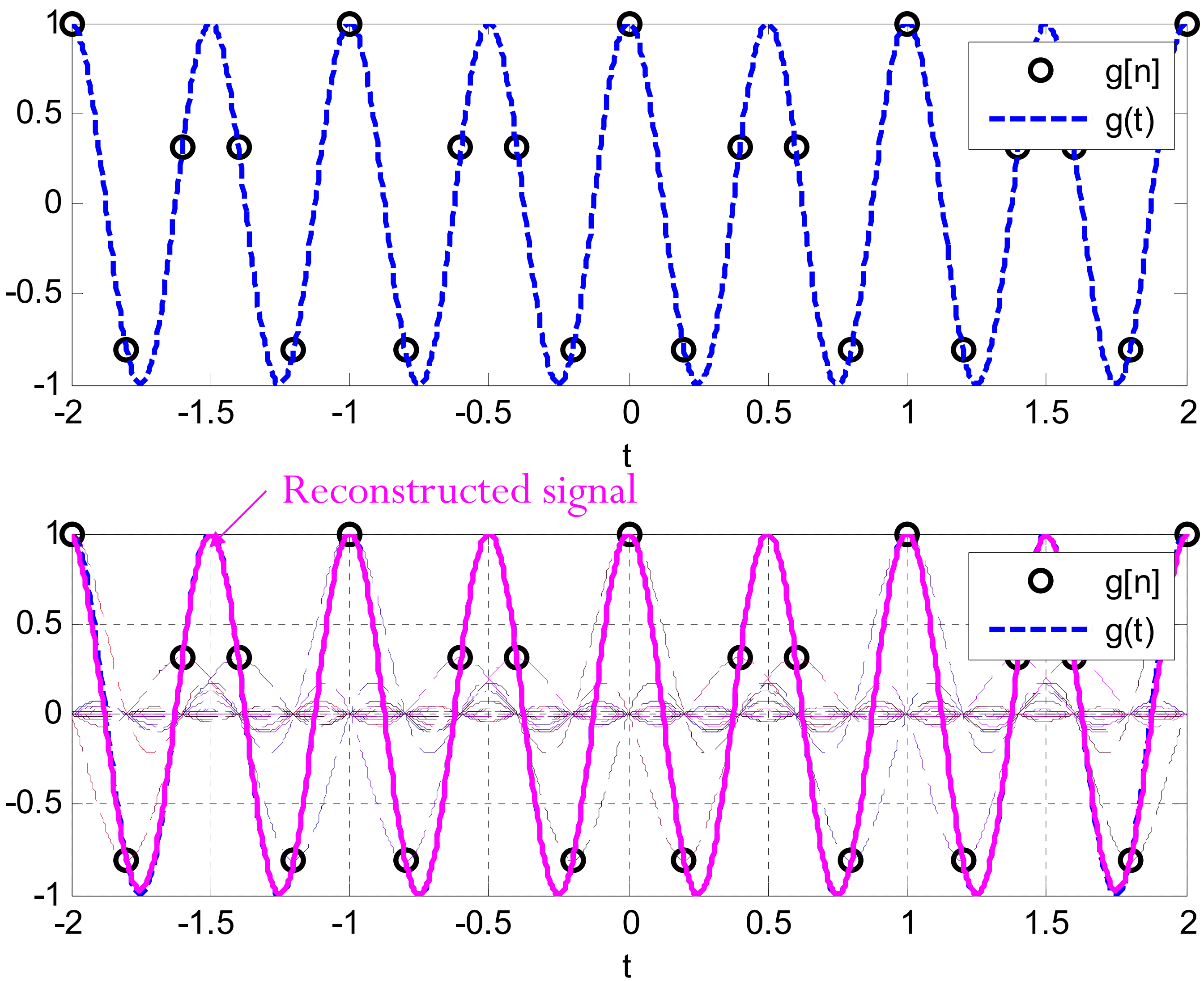
[Lower plot in Figure 50.]

Reconstruction of $\cos(2\pi(2)t)$

B = 2 Hz.

$T_s = 0.2$

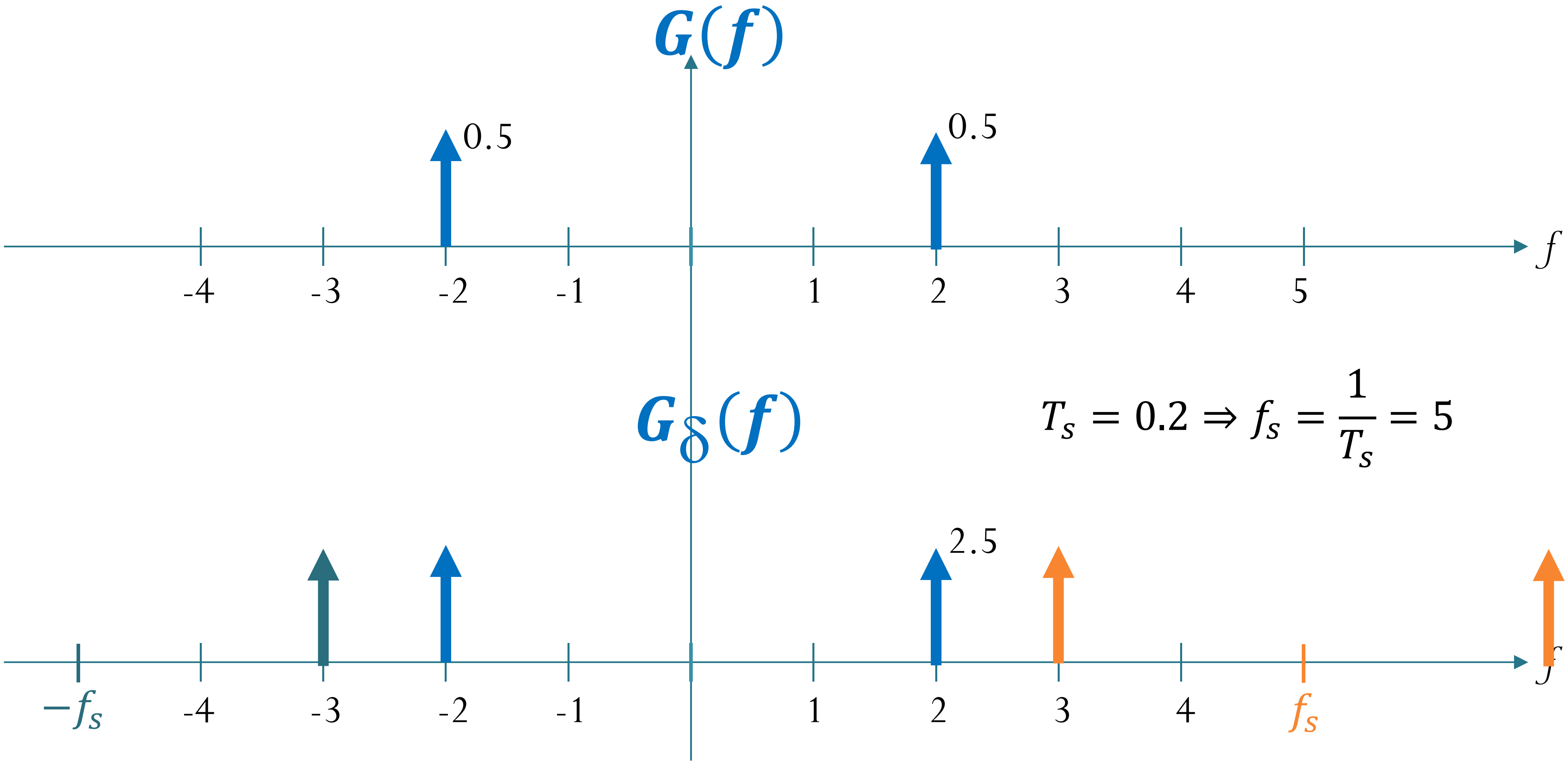
$f_s = \frac{1}{0.2} = 5$ [Sa/s]



[Lower plot in Figure 50.]

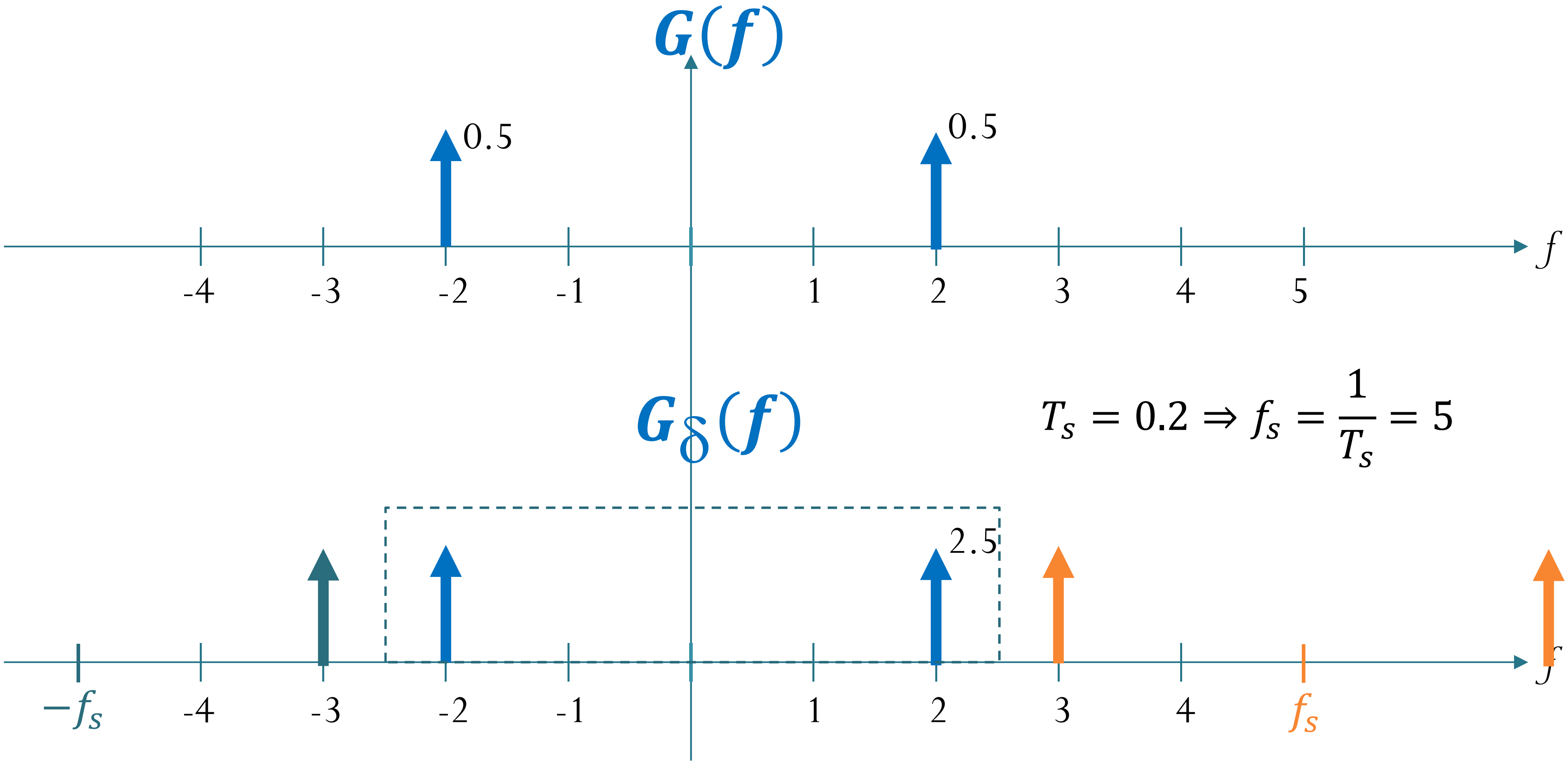
$f_s > 2B \Rightarrow$ the reconstructed signal is "the same" as the original signal.

$G_\delta(f)$ when $g(t) = \cos(2\pi(2)t)$



$$T_s = 0.2 \Rightarrow f_s = \frac{1}{T_s} = 5$$

Reconstruction of $g(t) = \cos(2\pi(2)t)$



$$T_s = 0.2 \Rightarrow f_s = \frac{1}{T_s} = 5$$

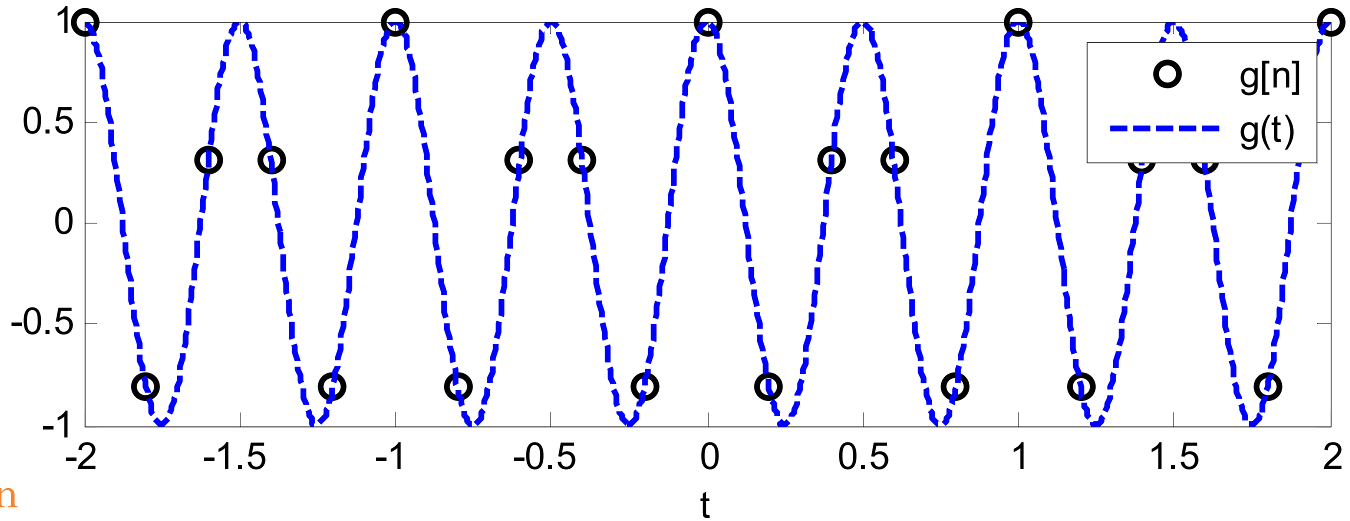
$$\hat{g}(t) = \cos(2\pi(2)t)$$

Reconstruction of $\cos(2\pi(2)t)$

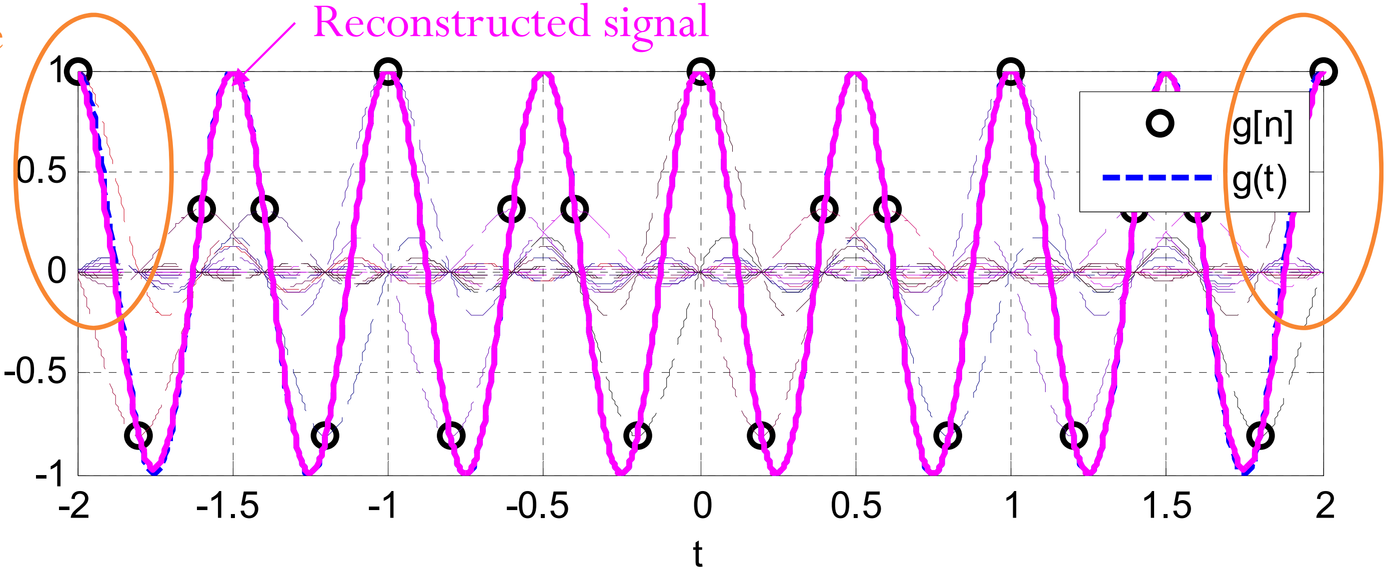
B = 2 Hz.

$T_s = 0.2$

$f_s = \frac{1}{0.2} = 5$ [Sa/s]



Some reconstruction error is visible at the boundaries because we did not use g[n] for n beyond ±2 in the reconstruction here.



$f_s > 2B \Rightarrow$ the reconstructed signal is "the same" as the original signal.

Triangular (linear) interpolation

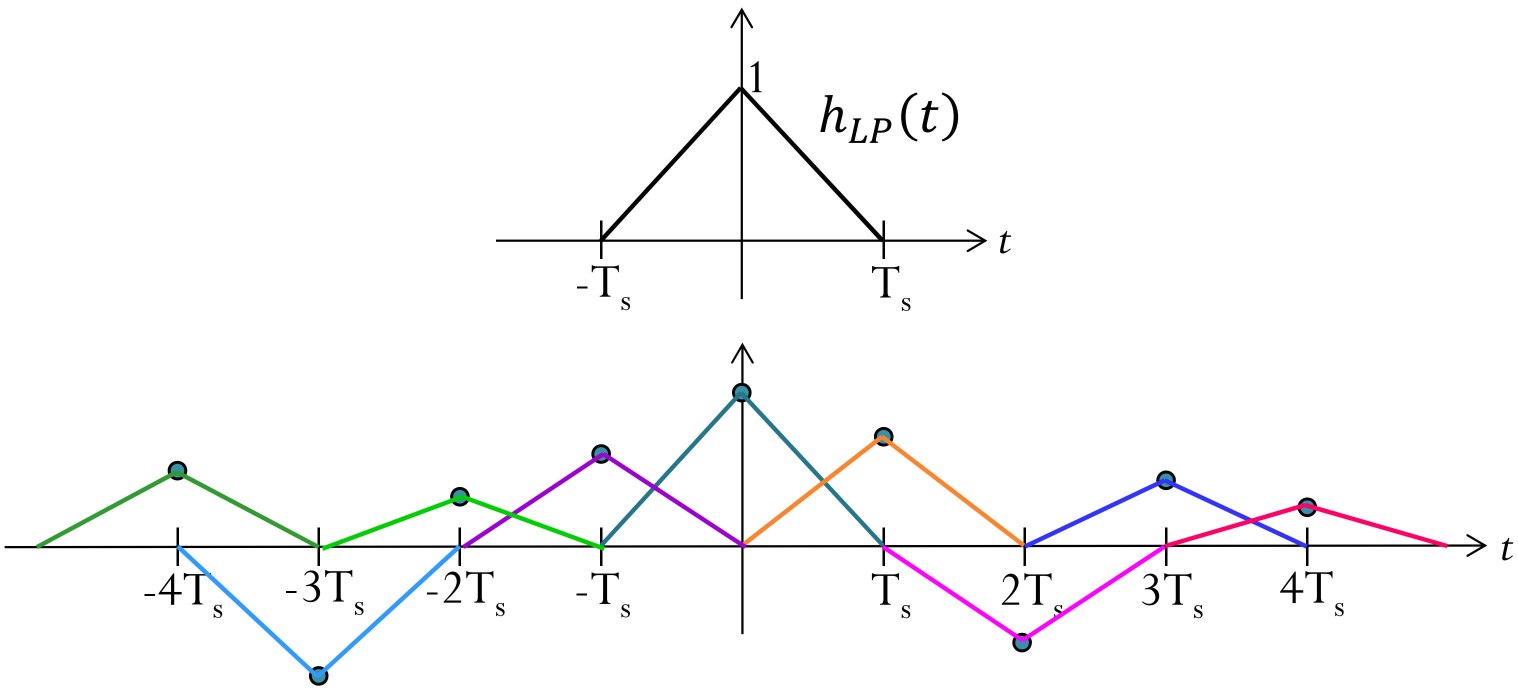


Figure 51

Triangular (linear) interpolation

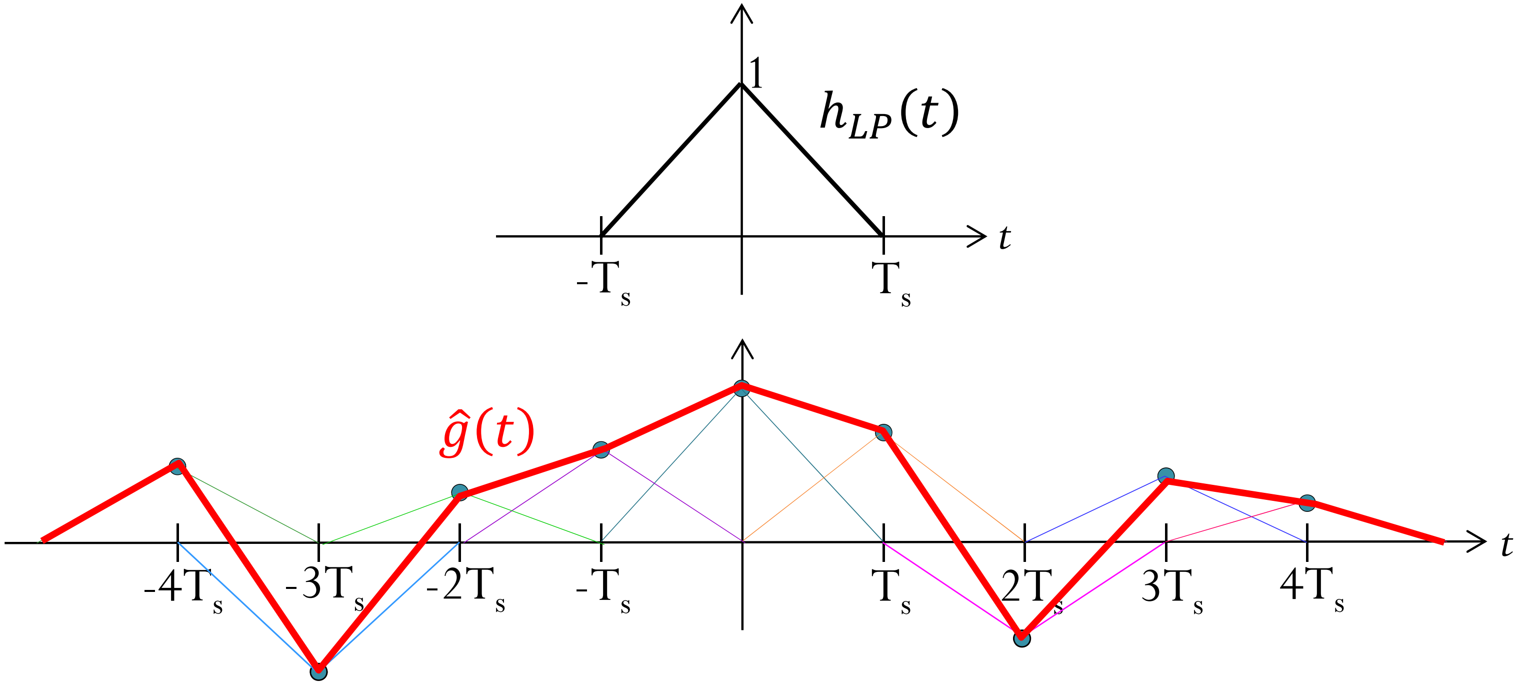


Figure 51

sinc vs. triangular interpolation

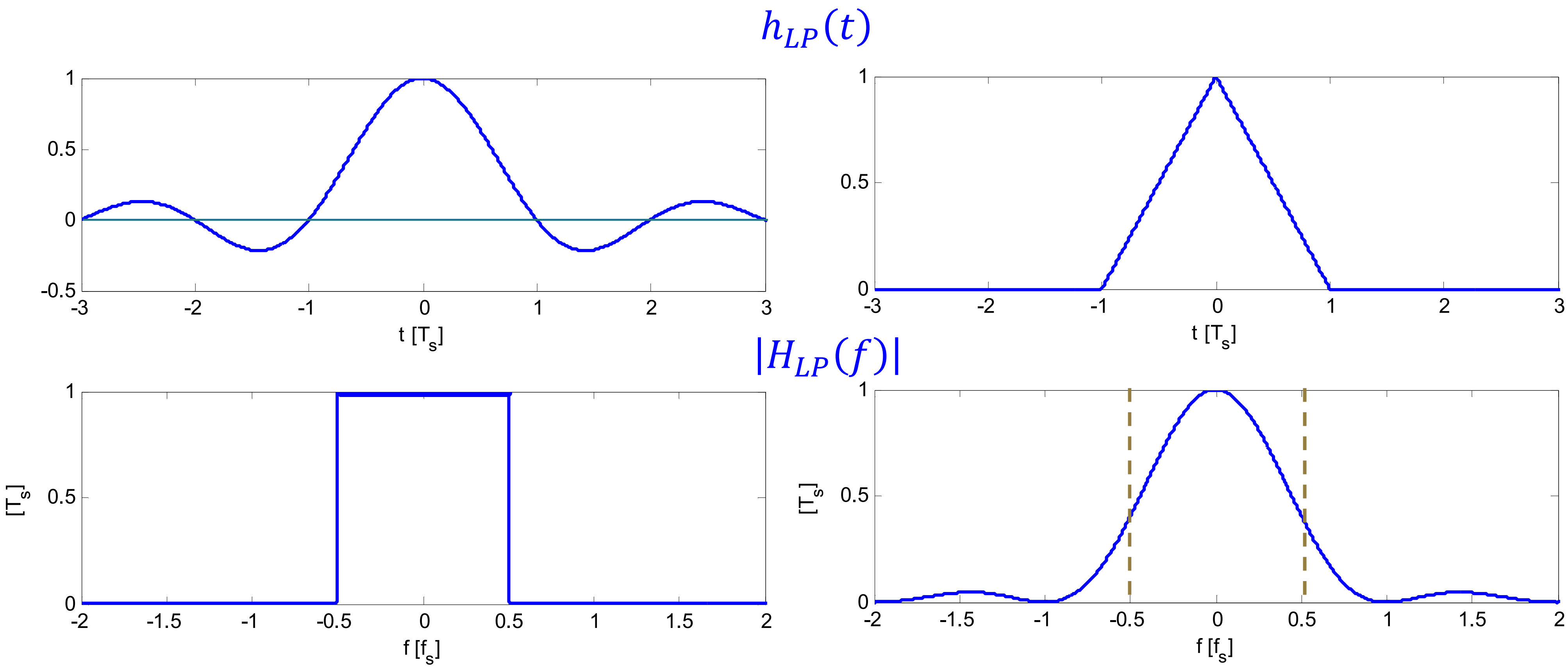


Figure 52

sinc vs. triangular interpolation

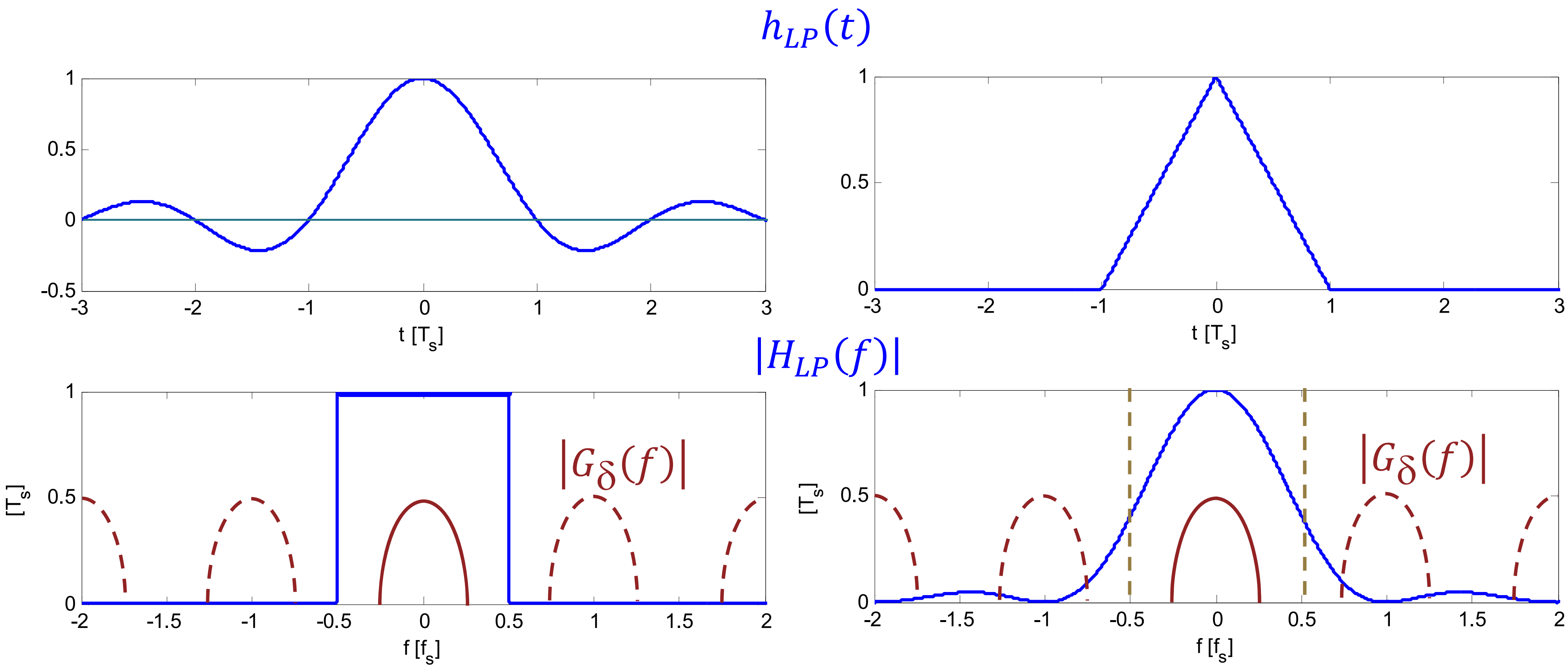
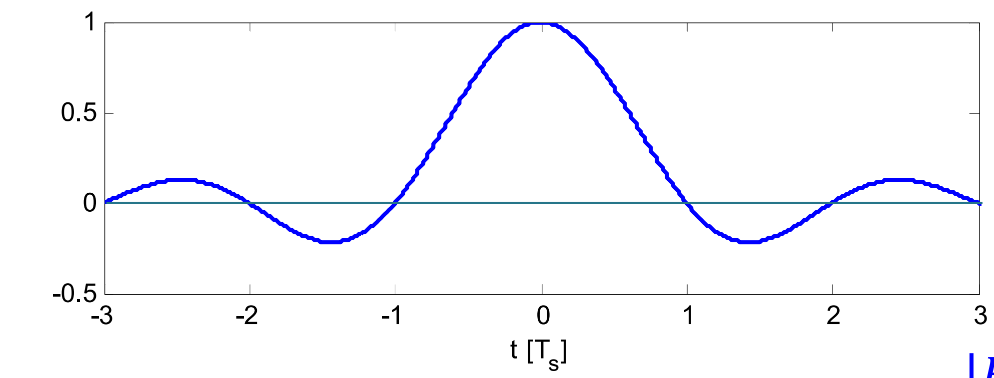
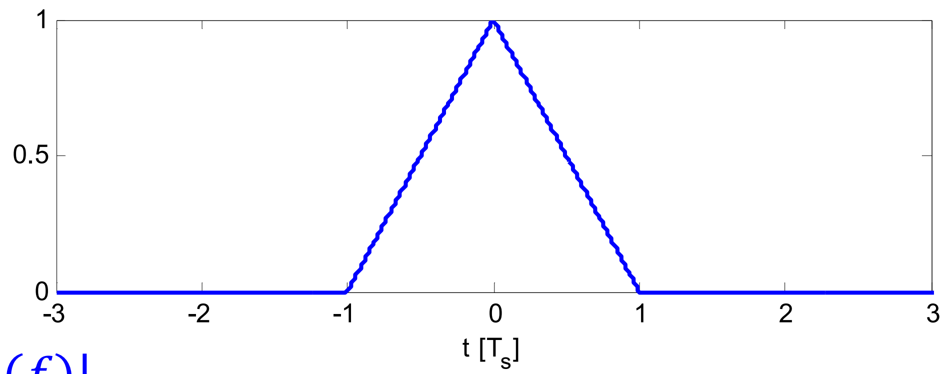


Figure 52

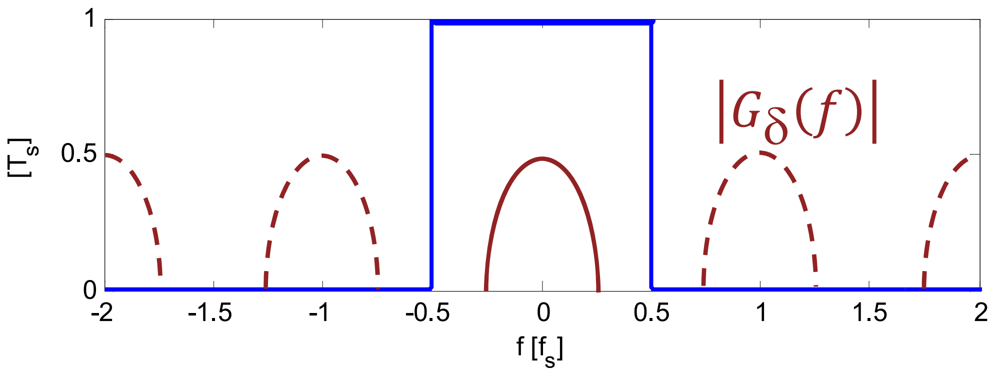
sinc vs. triangular interpolation

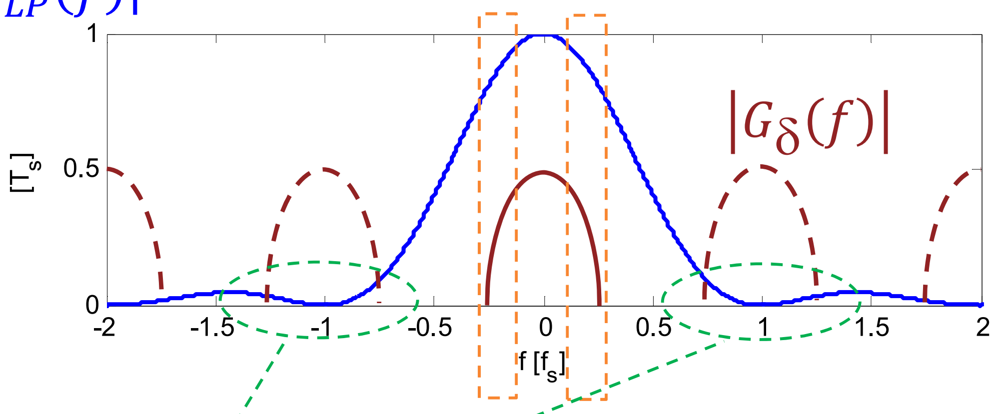
$h_{LP}(t)$





$|H_{LP}(f)|$





High freq. content
of G(f) is attenuated

(Small part of) the replicas at even higher
freq. (which do not exist before) also survive.